University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA30188: ALGEBRAIC CURVES

Tuesday 14th June 2005, 09.30–11.30

No calculators may be brought in and used.

Full marks will be given for correct answers to THREE questions. Only the best three answers will contribute towards the assessment.

Examiners will attach importance to the number of well-answered questions.

1. (a) What does it mean to say that an irreducible projective plane curve C is rational? What does it mean to say that a point of C is non-singular?

   (b) Find the singular points and the points at infinity on the complex projective plane curve given on the affine piece $z \neq 0$ by

$$y^3 - x^4 + x^3 = 0.$$

   (c) Show that this curve is rational.

2. (a) Explain briefly how to define a group law on a smooth plane cubic curve E. (You need not prove that the law you have defined is a group law.)

   (b) Say what it means for a point Q on a plane curve E to be an inflexion point.

   (c) From now on let k be the field $\mathbb{F}_{37}$ (the finite field with 37 elements). Take E to be the curve given in homogeneous coordinates on $\mathbb{P}^2_k$ by the equation

$$y^2z - x^3 + 9xz^2 - 11z^3 = 0$$

   and take the identity element to be the point $(0:1:0)$. Show that $P = (0:23:1)$ is a point of E. Find the point Q at which the tangent to E at P meets E again.

   (d) Show that Q is an inflexion point of E. Deduce that P is a point of order 6 in the group E.

   [In parts (c) and (d) you may find it useful to know that $22^2 \equiv 3 \bmod 37$ and $23^2 \equiv 11 \bmod 37$.]

3. (a) Explain carefully what is meant by a map $\phi\colon V \to W$ between two irreducible affine varieties over an algebraically closed field k. Define the corresponding map $\phi^*\colon k[W] \to k[V]$. What does it mean to say that ϕ is an isomorphism? What property does ϕ^* have in this case?

   (b) Let k be an algebraically closed field of characteristic $p > 0$. The Frobenius map $\Phi\colon k \to k$ is given by $a \mapsto a^p$ for all $a \in k$. Show that if $b \in k$ and $a^p = b$, then the polynomials $X^p - b$ and $(X - a)^p$ are equal. Deduce that Φ is bijective.

   (c) Say why the Frobenius map may also be thought of as a map of algebraic varieties $\Phi\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$. Is it an isomorphism of varieties? Justify your answer.

4. (a) Suppose B is a commutative ring and A is a finite B-algebra; that is, A is a commutative ring containing B as a subring and there are finitely many non-zero elements $a_1, \ldots, a_n \in A$ such that $A = Ba_1 + \cdots + Ba_n$. Suppose that I is a proper ideal of B (so $I \neq B$). Prove Nakayama's Lemma, which says that $IA \neq A$.

[Hint: if $IA = A$, each a_i may be written as a linear combination of the a_j with coefficients in I.]

(b) Let $V \subset \mathbb{A}_k^n$ be an affine variety over an algebraically closed field k. If the ring of polynomial functions on $\mathbb{A}_k^n$ is denoted by $k[X_1, \ldots, X_n]$, what is meant by the coordinate ring $k[V]$ of V?

(c) Describe the map $\phi\colon V \to \mathbb{A}_k^1$ corresponding to the map $\pi\colon k[X_1] \to k[V]$ given by $X_1 \mapsto X_1 + I(V)$.

(d) Suppose that the map π in part (c) is injective and makes $k[V]$ into a finite $k[X_1]$-algebra. Show that ϕ is surjective, by using the Nullstellensatz and Nakayama's Lemma applied to the ideal in $k[X_1]$ generated by an element of the form $X_1 - a$.