University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES
EXAMINATION

MA40188: ALGEBRAIC CURVES

Thursday 17th May 2007, 16.30–18.30

No calculators may be brought in and used.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. (a) Define what is meant by an *ideal* of a commutative ring R. Define what it means for an ideal I of R to be a *prime ideal*.

(b) Define what is meant by an *affine variety* over a field K. Say what it means for an affine variety V over K to be *irreducible*.

(c) Say what is meant by a *polynomial map* $f\colon V \to W$, where V and W are affine varieties. Explain how such a map induces a map $f^*\colon K[W] \to K[V]$. Show that if f^* is injective and V is irreducible, then W is irreducible.

(d) Now suppose $K = \mathbb{C}$ and let $W \subset \mathbb{A}^3$ be the variety given by the three equations $xz = y^2$, $x = yz$ and $y = z^2$. By considering the map $W \to \mathbb{A}^1$ given by the z-coordinate, or otherwise, show that W is isomorphic to $\mathbb{A}^1$. Deduce that W is irreducible.

(e) Is the variety $W' \subset \mathbb{A}^3$ given by the two equations $xz = y^2$ and $x = yz$ irreducible? Justify your answer briefly.

2. (a) Define the *function field* $K(V)$ of an irreducible projective variety V over a field K.

(b) Say what is meant by a *rational map* $f\colon V \dashrightarrow W$, where W is a projective variety. What does it mean to say that f is *regular* at a point $P \in V$? What does it mean to say that f is *dominating*? What does it mean to say that f is *birational*?

(c) What does it mean to say that an irreducible projective curve C over $\mathbb{C}$ is *rational*? What does it mean to say that an irreducible affine curve C_0 is rational?

(d) Let C_0 be the curve in $\mathbb{A}^3$ over $\mathbb{C}$ given by the equations $xz = y$ and $x = z^2(z-1)$. By considering the intersection of C_0 with planes containing the z-axis, show that C_0 is rational.

(e) Prove, using (d) or otherwise, that the curve $C_1 \subset \mathbb{A}^2$ given by the equation $x^4 = y^2(y - x)$ is rational.

3. (a) Define *affine space* $\mathbb{A}^n$ and *projective space* $\mathbb{P}^n$ of dimension n over a field K.

(b) Say what is meant by a *homogeneous polynomial* and what is meant by a *homogeneous ideal* of $K[x_0, \ldots, x_n]$.

(c) If I is a homogeneous ideal of $K[x_0, \ldots, x_n]$, explain how it defines a projective variety $V(I) \subset \mathbb{P}^n$.

(d) For an affine variety $V_0 \subset \mathbb{A}^n$, define the *projective closure* $V \subset \mathbb{P}^n$ of V_0 and explain what is meant by the *points at infinity*.

(e) What does it mean to say that a point P of a projective variety V is *singular*?

(f) If $K = \mathbb{C}$ and $V_0 \subset \mathbb{A}^2$ is given by the equation $x^4 + x^2y^2 - 3x^2 + 3y^2 + 8y + 6 = 0$, find V and its points at infinity. There are three singular points of V: find them. [You may find it useful to notice that the solutions to $(3 - 2x^2)(x^2 + 3)^2 = 16$ include $x = \pm 1$.]

4. Write a short essay on plane cubic curves, concentrating on the nonsingular case. You do not need to cover all the topics mentioned in the course, but you should say what the main features are and describe at least one of them in more detail.