University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA40188: ALGEBRAIC CURVES

May 2012

No calculators may be brought in and used.

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. Let K be a field.

   (a) Define the *affine space* $\mathbb{A}^n_K$ and the *projective space* $\mathbb{P}^n_K$ over K. [2]

   (b) Say what is meant by *homogeneous coordinates* $(x_0 : \ldots : x_n)$ on $\mathbb{P}^n_K$. [2]

   (c) If K is the finite field $\mathbb{F}_q$ with q elements, how many points do $\mathbb{A}^n_K$ and $\mathbb{P}^n_K$ have? [2]

   (d) What does it mean to say that $f \in K[X_0, \ldots, X_n]$ is a *homogeneous polynomial*? [1]

   (e) What does it mean to say that $I \subset K[X_0, \ldots, X_n]$ is a *homogeneous ideal*? [2]

   (f) Define what is meant by an *affine variety* and a *projective variety* over K, explaining briefly why these definitions make sense. [3]

   (g) Explain how an affine variety $V \subset \mathbb{A}^n_K$ gives rise to a projective variety $W \subset \mathbb{P}^n_K$, the projective closure of V. [4]

   (h) Find the projective closure of the affine plane curve over $K = \mathbb{C}$ given by

   $$X_1^3 + X_1^2 X_2 + X_1 X_2^2 + X_2^2 = 0$$

   and find the points at infinity. [4]

2. In this question K is an algebraically closed field and $A = K[X_1, \ldots, X_n]$.

   (a) Suppose that I is an ideal of A. Define the *radical* $\sqrt{I}$ of I and show that it is an ideal. [3]

   (b) State the Nullstellensatz. [2]

   (c) Assuming the Nullstellensatz, show that $I(V(I)) = \sqrt{I}$ for any ideal I of A. [6]

   (d) If $V \subset \mathbb{A}^n_K$ is an irreducible affine variety over K, define the *coordinate ring* $K[V]$ and the *function field* $K(V)$ of V, and say what it means for $f \in K(V)$ to be *regular* at a point $P \in V$. [4]

   (e) Show that, if V is an irreducible affine variety over K and $f \in K(V)$ is regular at every point of V, then $f \in K[V]$. [3]

   (f) Let V be the curve in $\mathbb{A}^2_K$ given by $xy + y^2 = x + y$, and let $f(x, y) = \dfrac{x}{y} \in K(V)$. Where is f regular? [2]

3. (a) Say what is meant by an *affine hypersurface* $V \subset \mathbb{A}^n_K$ and a *projective hypersurface* $W \subset \mathbb{P}^n$. [3]

(b) Say what it means for an affine hypersurface V to be *singular* at $P \in V$. [2]

(c) Show that if V is an irreducible hypersurface in $\mathbb{A}^n_K$ and K is an algebraically closed field (possibly of characteristic $p > 0$) then there exist points of V at which V is not singular. [6]

(d) Say what it means for a projective hypersurface W to be singular at $Q \in W$. [3]

(e) Find all the singular points of the projective curve $C \subset \mathbb{P}^2$ over $\mathbb{C}$ given by $x^3(x+z) - 2x^2yz - 2y^3z = 0$. [6]

4. (a) Define what it means for an irreducible projective variety W over a field K to be *rational*. [2]

(b) Show that the nonsingular plane cubic curve over K given by $y^2 = x(x-1)(x-a)$, with $a \neq 0, 1$, is not rational. You may assume that the characteristic of K is not 2. [9]

(c) Explain briefly why the nodal plane cubic curve is rational. You may assume that any two nodal plane cubics are isomorphic, so you may take the nodal plane cubic to be given by $y^2 = x^2(x+1)$. [3]

(d) Give another proof that the nodal plane cubic curve is rational, by considering the projection $\mathbb{A}^3 \to \mathbb{A}^2$ given by $(X, Y, Z) \mapsto (Y, X+Z)$ and restricting it to the twisted cubic curve $C = \{(t, t^2, t^3) \mid t \in K\} \subset \mathbb{A}^2$. [6]