University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES

MA40188

Algebraic curves

Friday 24th January 2014
16:30 – 18:30
2 hours

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

No calculators may be brought in and used.

PLEASE FILL IN THE DETAILS ON THE FRONT OF YOUR ANSWER BOOK/COVER AND SIGN IN THE SECTION ON THE RIGHT OF YOUR ANSWER BOOK/COVER, PEEL AWAY ADHESIVE STRIP AND SEAL.

TAKE CARE TO ENTER THE CORRECT CANDIDATE NUMBER AS DETAILED ON YOUR DESK LABEL.

DO NOT TURN OVER YOUR QUESTION PAPER UNTIL INSTRUCTED TO BY THE CHIEF INVIGILATOR.

1. Let $\mathbb{k}$ be a field and let $\mathbb{A}^n$ denote affine space of dimension n over $\mathbb{k}$.

(a) Let $J \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be an ideal and let $X \subseteq \mathbb{A}^n$ be a subset. Define the algebraic subset $\mathbb{V}(J)$ and the ideal $\mathbb{I}(X)$. [2]

(b) Define what it means for an algebraic subset $X \subseteq \mathbb{A}^n$ to be *irreducible*, and prove that X is irreducible if and only if $\mathbb{I}(X)$ is prime. [7]

(c) Consider the ideal $J = \langle x^2 - y^3, y^2 - z^3 \rangle$ in $\mathbb{k}[x, y, z]$.

(i) For $f \in \mathbb{k}[x, y, z]$, describe how to find $g \in J$ and $a, b, c, d \in \mathbb{k}[z]$ such that $f = g + a + bx + cy + dxy$. [4]

(ii) By using part (i) or otherwise, show that J is the kernel of the $\mathbb{k}$-algebra homomorphism $\phi\colon \mathbb{k}[x, y, z] \longrightarrow \mathbb{k}[t]$ satisfying [4]

$$\phi(x) = t^9; \quad \phi(y) = t^6; \quad \phi(z) = t^4.$$

(iii) Assume that $\mathbb{k}$ is algebraically closed. Using part (ii) or otherwise, prove that $\mathbb{V}(J)$ is irreducible. [3]

2. Let $\mathbb{k}$ be an infinite field, and let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic subsets and write $\mathbb{k}[x_1, \ldots, x_n]$ and $\mathbb{k}[y_1, \ldots, y_m]$ for the coordinate rings of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively.

(a) Define the coordinate ring $\mathbb{k}[X]$. [1]

(b) Show that $\mathbb{k}[X]$ is isomorphic as a $\mathbb{k}$-algebra to a quotient of a polynomial ring (you need not prove that your map is a $\mathbb{k}$-algebra homomorphism). [3]

(c) Define what it means for a map $\phi\colon X \to Y$ to be a *polynomial map*, and write down the definition of the *pullback* ϕ^* of ϕ. [3]

(d) Show that a $\mathbb{k}$-algebra homomorphism $\alpha\colon \mathbb{k}[Y] \to \mathbb{k}[X]$ determines a polynomial map $\phi\colon X \to Y$ such that $\phi^* = \alpha$. [8]

(e) Let $\mathbb{k}[x, y]$ denote the coordinate ring of $\mathbb{A}^2$. The following curves arise as the image of the given polynomial maps:

(i) $C_1 = \mathbb{V}(y^2 - x^3 - x^2)$ is the image of $\phi_1\colon \mathbb{A}^1 \to \mathbb{A}^2$ with $\phi_1(t) = (t^2 - 1, t^3 - t)$;

(ii) $C_2 = \mathbb{V}(y - x^5)$ is the image of $\phi_2\colon \mathbb{A}^1 \to \mathbb{A}^2$ with $\phi_2(t) = (t, t^5)$.

In each case, state whether the map induces an isomorphism between $\mathbb{A}^1$ and its image. Justify your response. [5]

3. Let $\mathbb{C}$ be the field of complex numbers, and let $\mathbb{C}[x_1, \ldots, x_n]$ denote the coordinate ring of $\mathbb{A}^n$.

(a) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be irreducible such that $f \notin \mathbb{C}$, and write $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$. Define the *tangent space* T_pX at a point $p \in X$. [2]

(b) Use the Nullstellensatz to prove that every $g \in \mathbb{C}[x_1, \ldots, x_n]$ that vanishes at each point of X is of the form $g = hf$ for some $h \in \mathbb{C}[x_1, \ldots, x_n]$. [4]

(c) What does it mean to say that p is a *singular* point of the hypersurface X from part (a)? Using part (b) or otherwise, prove that the set of non-singular points of X is a nonempty, Zariski-open subset. [7]

(d) Consider the following hypersurfaces:

(i) $X = \mathbb{V}(f) \subseteq \mathbb{A}^2$ for $f = y^2 - x^3 - x^2 \in \mathbb{C}[x, y]$;

(ii) $Y = \mathbb{V}(g) \subseteq \mathbb{A}^3$ for $g = (x-1)^2x^2 + x^2y^2 + z^2 \in \mathbb{C}[x, y, z]$.

In each case, determine the set of singular points on the hypersurface, explaining fully your conclusion. [7]

4. Let $\mathbb{k}$ be a subfield of the field of complex numbers $\mathbb{C}$.

(a) Define the *projective plane* $\mathbb{P}^2$ over $\mathbb{k}$, and explain briefly how to regard $\mathbb{P}^2$ as '$\mathbb{A}^2$ plus asymptotic directions', illustrating your understanding with a discussion of parallel lines in $\mathbb{A}^2$. [5]

(b) Let $C = \mathbb{V}(f) \subseteq \mathbb{P}^2$ be a nonsingular cubic curve for $f \in \mathbb{k}[x, y, z]$, and let $L \subseteq \mathbb{P}^2$ be a line. Show that $L \cap C$ comprises at most 3 points (equal to 3 in a suitable field extension of $\mathbb{k}$), and describe briefly how multiple intersections of L and C relate to geometric notions of tangency and inflection. [5]

(c) Suppose $C = \mathbb{V}(y^2z - x^3 - axz^2 - bz^3) \subseteq \mathbb{P}^2$ is nonsingular, and let $O = [0 : 1 : 0]$. Show how to construct a group law on C with O as identity element. Explain the construction of the inverse, and how your construction deals with multiple points of intersection. [You do need not prove the group axioms.] [5]

(d) Suppose C is given in affine coordinates by the equation $y^2 = x^3 + 4x$, and O is chosen to be the point at infinity on C.

(i) Interpret the condition $2p = O$ in the group law in geometric terms. [2]

(ii) Show that the tangent line to C at $p = (2, 4)$ passes through $(0, 0)$. Hence or otherwise, deduce that p is a point of order 4 in the group law. [3]