University of Bath

DEPARTMENT OF MATHEMATICAL SCIENCES EXAMINATION

MA40188: ALGEBRAIC CURVES

May 2010

ANSWERS

~~No calculators may be brought in and used.~~

Full marks will be given for correct answers to THREE questions.
Only the best three answers will contribute towards the assessment.

1. (a) R is a *noetherian* ring if every ideal is finitely generated [2, bookwork]

(b) Suppose J is an ideal of $R[t]$. Define I to be the set of all leading coefficients of polynomials in J, together with 0: this is an ideal in R. It is therefore fg, by $a_1, \ldots, a_N$ say. Choose $f_j \in J$ with leading coefficient a_j of degree d_j say, and suppose wlog that $d_1 \geq d_j$ for all j.
Given $g \in J$ with leading coefficient b, we may write $b = \sum \lambda_j a_j$ because $b \in I$, and if $\deg g = D \geq d_1$ we may consider

$$h = g - \sum \lambda_j t^{D-d_j} f_j$$

which is in J and has degree $< D$.
Now for each $d < D$ let I_d be the ideal (it is an ideal) of leading coefficients of polynomials in J of degree exactly d, together with 0. Again choose finitely many generators a_{id} for I_d and corresponding polynomials f_{id}. If h is of degree d we may reduce the degree of h by using the generators of I_d in exactly the same way as before (we don't need the power of t) and hence J is generated by the f_j together with all the f_{id}. [10, bookwork]

(c) Choose a minimal set of generators S for $I(V)$. If $f \in S$ then f is irreducible (because $I(V)$ is prime) so we put $W = V(f)$. By the conditions on V, either $W = \mathbb{A}^n$, which is impossible by the Nullstellensatz, or $W = V$, in which case $I(V)$ is generated by f. [8, unseen]

2. (a) A rational map $\phi\colon X \dashrightarrow Y$ is given by $m+1$ homogeneous elements $f_0,\ldots,f_m \in K[t_0,\ldots,t_n]$, all homogeneous of the same degree d, such that the point $(f_0(x) : \ldots : f_m(x)) \in \mathbb{P}^m$ is defined for some $x \in X$ (i.e. the $f_i(x)$ are not all zero) and is in Y for all $x \in X$ for which it is defined. We say that ϕ is a morphism if it is defined for every $x \in X$: note that the representation of ϕ as $(f_0 : \ldots : f_m)$ is not unique as we are free to multiply by or cancel any homogeneous polynomial factor and to add any homogeneous element of $I(X)$ of the right degree to any of the f_i, and $\phi(x)$ need only be defined by some of these representations. [5, bookwork]

(b) X and Y are birationally equivalent if there exist dominating rational maps $\phi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow X$ such that $\psi\circ\phi$ and $\phi\circ\psi$ are both the identity where they are defined. They are isomorphic if both ϕ and ψ may be chosen to be morphisms. [2, bookwork]

(c) σ is a morphism because it is defined unless $x_0y_0 = x_0y_1 = x_1y_0 = x_1y_1 = 0$: if that happens then from the first two either $y_0 = y_1 = 0$, which is not possible, or $x_0 = 0$; but then $x_1 \neq 0$ so from the last two $y_0 = y_1 = 0$ again. It is injective because $(x_0 : x_1) = (z_0 : z_2)$ and $(y_0 : y_1) = (z_2 : z_3)$. If $z_2 = 0$ we may instead take $(x_0 : x_1) = (z_1 : z_3)$ (note that $z_0z_3 = z_1z_2$): if both these fail then all the z_i vanish, and similarly for $(y_0 : y_1)$. [6, unseen]

(d) The inverse is given by $(z_1, z_2, z_3) \mapsto ((z_1 : z_3), (z_2 : z_3))$. The domain is given by the requirement that at least one of x_0y_1, x_1y_0 and x_1y_1 is nonzero: this fails if (and only if) $x_1 = y_1 = 0$, i.e. at $((1:0),(1:0))$. The image is the whole of $\mathbb{P}^2$: unless $z_3 = 0$ and either $z_1 = 0$ or $z_2 = 0$ we have already computed a preimage, but $(1:0:0) = \phi((1:0),(0:1))$ and $(0:1:0) = \phi((0:1),(1:0))$. [5, unseen]

(e) $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$ because in $\mathbb{P}^2$ any two curves meet and in $\mathbb{P}^1 \times \mathbb{P}^1$ lines in the same ruling do not meet. [2, bookwork]

3. (a) It may be assumed that K is algebraically closed. Suppose $\dim I_{P_1,\ldots,P_8}(3) \geq 3$. Let ℓ be the line through P_1 and P_2. Choose two more points on ℓ, say P_9 and P_{10}. The claim is that $\dim I_{P_1,\ldots,P_{10}}(3) \geq 1$. To see this, suppose that cubics C_1, C_2, C_3 pass through $P_1, \ldots, P_8$. Then $\lambda C_1 + \mu C_2 + \nu C_3$ passes through P_9 if and only if $\lambda C_1(P_9) + \mu C_2(P_9) + \nu C_3(P_9) = 0$, which is one linear condition on λ, μ, $\nu \in K$. So imposing the condition of passing through one more point drops the dimension by at most one, that is,

$$\dim I_{P_1\ldots P_8,P_9}(3) \geq \dim I_{P_1\ldots P_8}(3) - 1.$$

So there exists a nonzero cubic C passing through P_1, P_2, P_9 and P_{10} as well as $P_3, \ldots, P_8$. Hence $\ell \cap C \ni P_1$, P_2, P_9, P_{10} so $\ell \subset C$ (because a line meeting a cubic in more than three point is contained in it), so ℓ divides the equation of C by NSS. So $C = \ell Q$ for some conic Q. But $Q(P_3) = \cdots = Q(P_8) = 0$, since $C(P_i) = 0$ and $\ell(P_i) \neq 0$ for $i = 3, \ldots, 8$. This gives six points on the conic. [8, bookwork]

(b) It is enough to say that three collinear points add to zero. [3, bookwork]

(c) Put $f(x,y) = y^2 - x^3 - 11x - 3$ which has derivatives $-3x^2 - 11$ and $2y$, which evaluate to 8 and $15 = -16$ at P. The tangent line at P is therfore $-16(y+8) + 8(x-2) = 0$ and mutiplying by -2 gives $y = -15x - 9 = 16x - 9$. Therefore the line meets E again when $0 = -(16x-9)^2 + x^3 + 11x + 3$. The x^2 coefficient of this is $16^2 = 32 \times 8 = 8$ and that is the sum of the solutions: two of the solutions are 2 and 2 so the third one is 4. Taking $x = 4$ in the equation of the line gives $y = -15 \times 4 - 9 = 2 \times -30 - 9 = -7$. So $P + P = (4, 7)$ (note the change of sign).

The line through $(4,7)$ and $Q = (16, 11)$ is $3y = x + 17$, which after multiplying by -10 gives $y = -10x - 170 = -10x + 16$. So when we substitute in $f(x, y) = 0$ we get an x^2-coefficient of $100 = 7$ and the known solutions are $x = 4$ and $x = 16$ so the other solution is $x = 7 - 4 - 16 = -13 = 18$, and that gives $y = -9$ on the line so $(P + P) + Q = (18, 9)$.

The line through $P = (2, -8)$ and $Q = (16, 11)$ is $14y = 19x + 5$ which is $y = 8x + 7$ (use the hint) and so the x^2-coefficient is $64 = 2$ and the remaining x-coordinate is $2 - 2 - 16 = -16 = 15$: the y-coordinate on the line is $8 \times 15 + 7 = 3$ so $P + Q = (15, -3)$.

The line through $P = (2, -8)$ and $P + Q = (15, -3)$ is $13y = 5x + 10$ which (by the hint) gives $y = -2x - 4$, hence the x^2-coefficient is 4 so the remaining x-coordinate is $4 - 2 - 15 = -13 = 18$ and the y-coordinate on the line is $-2 \times 18 - 4 = -40 = -9$ so $(P + Q) + P = (18, 9)$. [9, unseen but similar on examples sheet]

4. (a) The tangent space $T_P V$ to a hypersurface $V = (f = 0) \subset \mathbb{A}^n$ in affine space at a point $P \in V$ is the subspace of K^n given by $\sum \frac{\partial f}{\partial x_i}(P) t_i = 0$. A singular point of V is one where $T_P V = K^n$, i.e. all partials vanish. [5, bookwork]

(b) Note that $\operatorname{Sing} V = V(f, \partial f/\partial x_i)$ so if $V = \operatorname{Sing} V$ then for each i we have $\partial f/\partial x_i \in \sqrt{I}(V) = I(V) = (f)$ by NSS, so f divides $(\partial f/\partial x_i)$. In characteristic zero this is impossible as $\deg f > \deg_{x_i} \partial f/\partial x_i > 0$. [5, bookwork]

(c) Take

$$f(x, y, z) = 4(x^2 + y^2 - xz)^3 - 27(x^2 + y^2)^2 z^2.$$

and put $z = 1$ to start with. Then we get

$$f(x, y) = 4(x^2 + y^2 - x)^3 - 27(x^2 + y^2)$$

and hence

$$f_x = 12(x^2 + y^2 - x)^2 \cdot (2x - 1) - 54(x^2 + y^2) \cdot 2x,$$
$$f_y = 12(x^2 + y^2 - x)^2 \cdot 2y - 54(x^2 + y^2) \cdot 2y.$$

On the singular locus the equation $f_y = 0$ gives $y = 0$ or $12(x^2 + y^2 - x)^2 - 54(x^2 + y^2) = 0$.

Let us do $y = 0$ first: then $f = 0$ gives $4(x^2 - x)^3 - 27x^4$ so $x = 0$ also or $4(x - 1)^3 = 27x$. And $(0, 0)$ also satisfies $f_x = 0$ so that's a singular point. If $x \neq 0$ we also have (for $y = 0$) the equation $f_x = 0$ which gives $(x - 1)^2(2x - 1) - 9x$. Now

$$27x = 4(x - 1)^3 = 3(x - 1)^2 \cdot (2x - 1)$$

but the last two are equal only if $x = 0$ or $x = -\frac{1}{2}$, and then the first equality fails. So we are left with $12(x^2 + y^2 - x)^2 - 54(x^2 + y^2) = 0$. Substituting for $54(x^2 + y^2)$ in $f_x = 0$ we get $x^2 + y^2 - x = 0$, but from $f = 0$ that also gives $x^2 + y^2 = 0$ so we just get $x = y = 0$ again.

So the only singular point with $z = 1$ is $(0 : 0 : 1)$. What about $z = 0$? There the equation of the curve becomes $4(x^2 + y^2)^3 = 0$ so the only points at infinity are $(1 : i : 0)$ and $(1 : -i : 0)$. It remains to see whether they are singular or not. We do this in the affine piece $x = 1$, where the curve has equation

$$4(1 + y^2 - z)^3 - 27(1 + y^2)z^2 = 0.$$

It is easy to see that this and both its first derivatives vanish at $(1, \pm i)$ so these are singular points.

Hence the singular locus is $\{(0 : 0 : 1), (1 : i : 0), (1 : -i : 0)\}$. [10, unseen]