MA40188 ALGEBRAIC CURVES
2015/16 SEMESTER 1
EXAM FEEDBACK

In general, this exam is comprehensive. It covers all main topics of the course. Each question consists of 5 parts, among which the first 4 parts are fairly standard, whose answers can be found in lecture notes or solutions to exercise sheets. Given the relatively large size of the class, the range of marks is also quite wide, including some very strong scripts and some very weak ones. Some common mistakes in the exam are as follows.

Question 1.

In parts (a) and (b), the definitions and the proof of $\mathbb{V} - \mathbb{I}$ correspondence is standard. Part (d) is the example of the twisted cubic, which can be found in exercise sheets for two different weeks. Part (e) boils down to proving the polynomial $y^2 - f(x)$ is irreducible when the degree of $f(x)$ is odd, whose proof appeared for several times in exercises.

The biggest common mistake is in part (c). Many people lost marks in proving the ideal $I = (x_1 - a_1, \cdots, x_n - a_n)$ is maximal. There are at least two different correct proofs in the exam scripts, one using the fundamental isomorphism theorem (as in the solution to the exercise sheet), the other using the definition of the maximal ideal (by proving that any ideal strictly larger than I must contain a constant function). However, some of you claimed that "$\mathbb{V}(I)$ is a point $\Longrightarrow I$ is a maximal ideal", which is a wrong statement if you don't know I is a radical ideal in the first place.

Question 2.

In parts (a) and (b), the definitions and the proof of $\mathbb{I}(X)$ being homogeneous is standard. In part (c), the rational map is indeed dominant, as it hits every point in $\mathbb{P}^2$ with all three coordinates nonzero. This is where many people lost marks. Part (d) is again a standard example. However, some of you have found it difficult because the formulas are not explicitly given.

Part (e) is a purely conceptual question. Many of you observed that $x, y^2, z^3 \in I$. With a little more effort we can prove $I = (x, y^2, z^3)$. This immediately implies I is a homogeneous ideal as it is generated by finitely many homogeneous polynomials. The projective algebraic set $\mathbb{V}(I)$ is the empty set since the only solution $x = y = z = 0$ doesn't give a point in $\mathbb{P}^2$.

Question 3.

This is the most popular question. The definitions in parts (a) and (b), as well as the proof of nonemptyness of nonsingular points in part (c), are standard. Part (d) is an example we did in a certain lecture. To find all singular points you need to solve the system of equations given by the original polynomial and its derivatives. Unfortunately there are lots of mistakes in computing the partial derivatives.

Part (e) is new, but similar to some questions in exercise sheets. To find singular points on a projective variety, the easiest way is to do the complete computation on one affine piece, then check the points at infinity individually on other affine pieces. In this question, one affine piece is already given. From the two derivatives, we can find that the only potential singular points are $(\pm\sqrt{a/3}, 0)$. Just plug in to the defining equation and simplify to get $4a^3 + 27b^2 = 0$. The only point at infinity $[0 : 1 : 0]$ is always a non-singular point, which can be checked on another affine piece.

Question 4.

This is the least popular question, but also the best answered one. The statement of the group law in part (a), the rationality in part (c) and Bézout's theorem in part (d) are all standard. Part (b) is a calculcation we did in a certain lecture. You can use either the projective coordinates or the affine coordinates to do the calculation.

Part (e) is new and interesting. There are various ways to prove it, but the key is to make full use of the fact that O is an inflection point. It implies that $\overline{O} = O$ hence $-P$ is the third intersection point of OP and the curve C. If you draw a picture here, it is easy to see that $P + P = -P$ is equivalent to the third intersection point of $T_P C$ and C being P again.