MA40188 ALGEBRAIC CURVES 2015/16 SEMESTER 1
EXAM INFORMATION
(CHRISTMAS EDITION)

1. Distribution of Questions

As you have experienced in many other exams, there will be four questions in the exam, each worth 20 marks. Your best three answers contribute towards the assessment. The following chart shows the material covered in each question.

Question number	Lectures covered
Question 1	Weeks 1-3
Question 2	Weeks 4-5
Question 3	Weeks 6-7
Question 4	Weeks 8-9

Week 10 lectures will not be examined.

2. Types of Questions

Each question consists of 5 parts, each worth 4 marks. They correspond to the following 5 types, one part of each type. But they are not necessarily listed in this order.

2.1. **Definitions and statements of results.** Definitions are always the building blocks in any branch of mathematics. Make sure you know the precise definitions of the mathematical concepts we learned in this unit. Moreover, it is also necessary to know the precise statements of all constructions, lemmas, propositions, theorems and corollaries, and understand their meaning. You will be asked to state some definitions, constructions or results. Here are some sample questions:

- What is an affine space? What is an affine algebraic set?
- State Hilbert's Nullstellensatz.
- State the group law on a non-singular cubic curve C with the identity element O.

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2.2. **Proofs of main results.** Understanding the proofs of important results is usually very helpful for understanding the results themselves. You will be asked to reproduce some proofs that we did in lectures. Some of the proofs are very short, while others are longer, in which case you could be asked to give a certain step in it. The best strategy for learning a proof is to think it through and make sure you understand a few key steps, then try to write down the whole proof in your own words. See Appledix A for a list of examinable proofs. Here are some sample questions:

- Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set such that $\varnothing \neq X \neq \mathbb{A}^n$. Prove that X is the intersection of finitely many hypersurfaces.
- Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set. Show that the ideal $\mathbb{I}(X)$ is prime.

2.3. **Explicit examples.** To fully understand any concept or result, it is crucial to test it in plenty of examples. In this course we have given a lot of emphasis on examples. It is important that you understand all examples that we did in lectures and exercise sheets. The examples you will face in the exam will be very similar (or even identical) to some of them. Here are some sample questions:

- Consider the projective algebraic set $C = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$. Explain why $\varphi : \mathbb{P}^1 \to C$ defined by $\varphi([u : v]) = [u^2 : uv : v^2]$ is a morphism.
- Find all standard affine pieces of the projective variety $X = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$.
- Find all singular points in the affine variety $X = \mathbb{V}(x^3 + y^3 - 3xy) \subseteq \mathbb{A}^2$.

2.4. **Combination of the above types.** This part could be statements, or proofs, or examples, or a combination of of them. Here are some sample combination questions:

- What is a Noetherian ring? Briefly explain why $\Bbbk[x]$ is a Noetherian ring.
- For any subset $X \subseteq \mathbb{A}^n$, define its ideal $\mathbb{I}(X)$. Prove that $\mathbb{I}(X)$ is a radical ideal in $\Bbbk[x_1, \cdots, x_n]$.
- What does it mean by saying an affine algebraic set is irreducible? Give an example of an affine algebraic set which is reducible.

2.5. **Mysterious part.** The last part in each question is a mysterious part. You may or may not have seen it in lectures or exercises, but the techniques required to answer it should all be familiar. If you do not immediately see what to do, try to think which results or examples you have seen in lectures or exercises are likely to be related to the question at hand, and how you can apply them in the question. Even if you cannot solve such a problem completely, you should still write down what you have done to earn partial marks. In the end, these questions are not difficult, so please do not be afraid!

3. Miscellaneous

In the exam, you always need to fully justify your answers by showing all your reasonings and calculations. It will be very helpful if you can write clearly, and leave some empty space between lines and at the boundary of your answer sheets for marking purpose.

About 80% of exam questions (everything except the mysterious part) are taken from lecture notes or exercise sheets (either exactly the same, or very similar). So it is always a good idea to go through the lecture notes and exercises very carefully during your revision. A complete set of lecture notes, exercise sheets and solutions (combined in one file) is available to download from the course webpage.

The mysterious part will probably be something new to you, but not difficult. If you find yourself running into some extremely complicated arguments or calculations, it might be a good idea to look back to check if you have done anything wrong.

Please don't hesitate to contact me if you need any help during your revision. There are at least three ways to reach me:

- Come to the Q&A session or office hours (see details on the course webpage).
- Send your questions to me by email, which should work 24/7. I will respond at my first convenience.
- If you need to talk to me face to face outside of office hours, feel free to send me an email to request an appointment.

The exam should be a *happy ending* of the course. I hope everyone has learned something interesting from this course, and I wish you all the best of luck in the exam.

Appendix A. List of examinable proofs

In fact we didn't do too many proofs in this course. The following is a list of proofs that could be asked in the exam. Other proofs of lemmas, propositions, theorems and corollaries that we did in lectures are non-examinable. But you always need to know the statements of all results mentioned in lectures, regardless of whether their proofs are examinable or not.

- Proposition 1.7: basic properties of affine algebraic sets.
- Lemma 1.10: affine algebraic sets defined by ideals.
- Corollary 1.14: polynomial rings over fields are Noetherian rings.
- Theorem 1.18: a non-trivial algebraic set is a finite intersection of hypersurfaces.
- Lemma 2.2: the radical of an ideal is an ideal.

- Lemma 2.6: properties of $\mathbb{I}$ in the affine situation.
- Proposition 2.9: the affine $\mathbb{V} - \mathbb{I}$ correspondence.
- Proposition 2.15: prime ideals and irreducible algebraic sets.
- lemma 4.17: projective algebraic sets defined by homogeneous polynomials.
- Lemma 4.22: property of $\mathbb{I}$ in the projective situation.
- Lemma 5.4: prime and radical principal ideals.
- Lemma 5.16: criterion for dominance.
- Proposition 6.1: affine charts of projective algebraic sets.
- Theorem 7.4: existence of non-singular points on an irreducible hypersurface.
- Proposition 8.7: rationality of lines and irreducible conics.
- Theorem 8.8: special case of Bézout's theorem.
- Theorem 8.12: special case of Bézout's theorem.
- Proposition 8.18: rationality of singular cubics.
- Proposition 9.8: commutativity, identity and inverse in the group law.
- Proposition 9.13: associativity in the group law.

Appendix B. List of Non-Examinable Exercises

Some exercises will show up in the exam. However, hints or intermediate steps are rarely given in the exam. In case that an exercise is very long, a certain part or step of it could still be a reasonable question in the exam.

The following exercises are non-examinable.

- Sheet 1: Ex 1.4.
- Sheet 2: Ex 2.1 (3).
- Sheet 4: Ex 4.1 (1), Ex 4.2 (2).
- Sheet 5: Ex 5.4.
- Sheet 6: Ex 6.3.
- Sheet 9: Ex 9.3.
- Sheet 10: Ex 10.1, Ex 10.2, Ex 10.3, Ex 10.4.