Exercise Sheet 1

This sheet will be discussed in the exercise class on 9 October. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 1.1. *Examples of algebraic sets.* For each of the following $X \subseteq \mathbb{A}^2$, find a set of polynomials $S \subseteq \Bbbk[x, y]$ such that $X = \mathbb{V}(S)$. You don't need to justify your answer.

(1) $X = \{(0,0), (0,1), (1,0), (1,1)\}$.

(2) $X = \{(0,0), (1,1)\}$.

(3) X is the union of the x-axis and a single point $(0, 1)$.

(4) For fun: describe the algebraic set $\mathbb{V}(xy, yz, zx) \subseteq \mathbb{A}^3$ geometrically.

Exercise 1.2. *Prove Proposition 1.7.* Consider algebraic sets in $\mathbb{A}^n$.

(1) Suppose $S_1 \supseteq S_2$. Prove that $\mathbb{V}(S_1) \subseteq \mathbb{V}(S_2)$.

(2) Prove that $\varnothing$ and $\mathbb{A}^n$ are algebraic sets in $\mathbb{A}^n$.

(3) Prove that $\cap_\alpha(\mathbb{V}(S_\alpha)) = \mathbb{V}(\cup_\alpha S_\alpha)$.

(4) Suppose $S = \{fg \mid f \in S_1, g \in S_2\}$. Prove that $\mathbb{V}(S_1) \cup \mathbb{V}(S_2) = \mathbb{V}(S)$. Use induction to conclude that the union of finitely many algebraic sets is still algebraic.

Exercise 1.3. *Examples of algebraic sets.* Prove that algebraic sets in $\mathbb{A}^1$ are just the finite subsets in $\mathbb{A}^1$ (including $\varnothing$) together with $\mathbb{A}^1$ itself. You can follow these steps:

(1) Verify that they are indeed algebraic sets.

(2) Prove that if an algebraic set in $\mathbb{A}^1$ is not $\mathbb{A}^1$ itself, then it contains at most finitely many points. (*Hint:* you can use the following lemma in algebra: a non-zero polynomial $f(x) \in \Bbbk[x]$ of degree d has at most d roots.)

(3) As an application of this exercise, give an example of infinitely many algebraic sets, whose union is not an algebraic set.

Exercise 1.4. *Prove Proposition 1.16.* Prove that if R is a Noetherian ring, then R/I is also Noetherian for any ideal I in R. You can follow these steps:

(1) We write the quotient ring homomorphism $q : R \to R/I$ (sending each $r \in R$ to the coset $r + I$). For any ideal J in R/I, prove that $q^{-1}(J)$ is an ideal in R.

(2) For two ideals $J_1 \subseteq J_2$ in R/I, prove that $q^{-1}(J_1) \subseteq q^{-1}(J_2)$.

(3) Suppose $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ is an ascending chain of ideals in R/I. Use (1), (2) and the fact that R is Noetherian to show that this chain stabilises.

(4) Use Proposition 1.15 to conclude that R/I is Noetherian.