Exercise Sheet 2

This sheet will be discussed in the exercise class on 16 October. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 2.1. *Some proofs in lectures.* We prove Lemma 2.2 and Proposition 2.12 (2).

(1) Let I be an ideal in a ring R. If $a^m \in I$ and $b^n \in I$ for some $a, b \in R$ and $m, n \in \mathbb{Z}_+$, show that $(a+b)^{m+n} \in I$. (*Hint:* use the binomial expansion.)

(2) Let I be an ideal in a ring R. Prove that $\sqrt{I}$ is an ideal and $I \subseteq \sqrt{I}$.

(3) Show that every maximal ideal is prime, and every prime ideal is radical.

Exercise 2.2. *Examples of radical and prime ideals.* Suppose a non-zero polynomial $f \in \Bbbk[x_1, \cdots, x_n]$ is factor as $f = uf_1^{k_1} \cdots f_t^{k_t}$ for some $0 \neq u \in \Bbbk$, $k_1, \cdots, k_t \in \mathbb{Z}_+$, and irreducible polynomials $f_1, \cdots, f_t$ which are pairwisely coprime.

(1) Show that (f) is a prime ideal if and only if f is an irreducible polynomial.

(2) Let $\overline{f} = f_1 \cdots f_t$. Show that $\sqrt{(f)} = (\overline{f})$. (*Remark:* this justifies Example 2.3.)

(3) Conclude that (f) is a radical ideal if and only if f has no repeated factors.

Exercise 2.3. *Examples of maximal ideals.* Find all maximal ideals in $\Bbbk[x_1, \cdots, x_n]$. You can follow these steps:

(1) For any fixed point $p = (a_1, \cdots, a_n) \in \mathbb{A}^n$, consider the ring homomorphism $\varphi_p : \Bbbk[x_1, \cdots, x_n] \to \Bbbk; f(x_1, \cdots, x_n) \mapsto f(a_1, \cdots, a_n)$. Show that $m_p := \ker(\varphi_p) = (x_1 - a_1, \cdots, x_n - a_n)$. Use Proposition 2.12 to show that m_p is a maximal ideal.

(2) What is $\mathbb{V}(m_p)$? Use Proposition 2.16 to show that every maximal ideal in $\Bbbk[x_1, \cdots, x_n]$ is of the form m_p for some $p \in \mathbb{A}^n$. (*Remark:* historically, this was proved before Nullstellensatz was established.)

Exercise 2.4. *A famous example: the twisted cubic.* Prove that the subset in $\mathbb{A}^3$ given by $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in \Bbbk\}$ is an affine variety. You can follow these steps:

(1) Show that X is the algebraic set $\mathbb{V}(I)$ for the ideal $I = (y - x^2, z - x^3) \subseteq \Bbbk[x, y, z]$.

(2) Show that $\Bbbk[x, y, z]/I \cong \Bbbk[t]$.

(3) Use Proposition 2.12 to conclude that I is a prime ideal, hence a radical ideal. Use Proposition 2.9 to conclude that $I = \mathbb{I}(X)$. Use Proposition 2.15 to conclude that X is an affine variety. (*Remark:* X is called the affine *twisted cubic.*)

(*Remark:* Exercise 3.4 will be a continuation of this one.)