## Exercise Sheet 3

*This sheet will be discussed in the exercise class on 23 October. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.*

**Exercise 3.1.** *Example: the graph of a polynomial function.*

(1) Show that the *projection map* $\pi : \mathbb{A}^n \to \mathbb{A}^r$, $n \geqslant r$, defined by $\pi(a_1, \cdots, a_n) = (a_1, \cdots, a_r)$ is a polynomial map.

(2) Let $X \subseteq \mathbb{A}^n$ be an algebraic set and $f \in \Bbbk[X]$. Define the subset $G(f) \subseteq \mathbb{A}^{n+1}$ by $G(f) = \{(a_1, \cdots, a_n, a_{n+1}) \in \mathbb{A}^{n+1} \mid (a_1, \cdots, a_n) \in X \text{ and } a_{n+1} = f(a_1, \cdots, a_n)\}$. Show that $G(f)$ is an algebraic set. (*Remark:* $G(f)$ is called the *graph* of $f$.)

(3) Show that the map $\varphi : X \to G(f)$; $(a_1, \cdots, a_n) \mapsto (a_1, \cdots, a_n, f(a_1, \cdots, a_n))$ is a polynomial map.

(4) Show that $\varphi$ is an isomorphism of algebraic sets by writing down the inverse polynomial map $\psi : G(f) \to X$, and checking both compositions are identities.

(5) Briefly explain why Example 3.14 is a special case of this.

**Exercise 3.2.** *Example: a nodal cubic.* Consider $X = \mathbb{V}(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$.

(1) Show that $\varphi : \mathbb{A}^1 \to X$; $t \mapsto (t^2 - 1, t^3 - t)$ is a polynomial map.

(2) Show that $\varphi$ is surjective but not injective on points, hence not an isomorphism.

(3) Show that $y^2 - x^3 - x^2$ is an irreducible polynomial. Use Exercise 2.2 to conclude that $I = (y^2 - x^3 - x^2)$ is a prime ideal, hence radical. Use Propositions 2.15 and 2.9 to conclude that $X$ is an affine variety and $\mathbb{I}(X) = I$.

**Exercise 3.3.** *Example: a cuspidal cubic.* Consider $X = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$.

(1) Show that $\varphi : \mathbb{A}^1 \to X$; $t \mapsto (t^2, t^3)$ is a polynomial map.

(2) Show that $\varphi$ is injective and surjective on points.

(3) Show that $y^2 - x^3$ is an irreducible polynomial. Conclude that $I = (y^2 - x^3)$ is a prime ideal, hence radical. Conclude that $X$ is an affine variety and $\mathbb{I}(X) = I$.

(4) Show that the $\Bbbk$-algebra homomorphism $\varphi^* : \Bbbk[X] \to \Bbbk[\mathbb{A}^1]$ is not surjective. Conclude by Proposition 3.22 that $\varphi$ is not an isomorphism.

**Exercise 3.4.** *Example: the twisted cubic, revisited.* This is a continuation of Exercise 2.4. We consider the polynomial map $\varphi : \mathbb{A}^1 \to X$; $t \mapsto (t, t^2, t^3)$.

(1) Show that $\varphi$ is an isomorphism by writing down the inverse $\psi : X \to \mathbb{A}^1$ and computing the two compositions.

(2) Show that $\varphi$ is an isomorphism by proving that $\varphi^*$ is an isomorphism.