Exercise Sheet 4

This sheet will be discussed in the exercise class on 30 October. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 4.1. *Get familiar with projective spaces.* Answer the following quick questions.

(1) What is $\mathbb{P}^0$? Why does $\mathbb{P}^1$ have only one more point than $\mathbb{A}^1$? When $\Bbbk = \mathbb{C}$, can you picture $\mathbb{P}^1_{\mathbb{C}}$ as a bubble (or a ball, something like that)? Which points in $\mathbb{P}^n$ belong to only one of the U_i's in the standard affine cover of $\mathbb{P}^n$?

(2) Follow Example 4.9 to find the points at infinity for the affine algebraic set $\mathbb{V}_a(x_2^2 - x_1^2 - 1) \subseteq \mathbb{A}^2$. Do the same for $\mathbb{V}_a(x_2^2 - x_1^2)$ and $\mathbb{V}_a(x_2^2 - x_1^3)$ in $\mathbb{A}^2$.

Exercise 4.2. *Properties of homogeneous polynomials and ideals.*

(1) Let $f \in \Bbbk[z_0, \cdots, z_n]$ be a non-zero homogeneous polynomial. Assume $f = gh$ for some $g, h \in \Bbbk[z_0, \cdots, z_n]$. Show that g and h are also homogeneous polynomials.

(2) Show that an ideal $I \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$ is homogeneous if and only if it can be generated by a finite set of homogeneous polynomials.

(3) Suppose a homogeneous ideal $I \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$ is generated by a finite set of homogeneous polynomials $S = \{f_1, \cdots, f_m\}$. Show that $\mathbb{V}_p(I) = \mathbb{V}_p(S)$.

Exercise 4.3. *Projective spaces are better than affine spaces!* A *line* in $\mathbb{P}^2$ is a projective algebraic set $\mathbb{V}_p(f)$ defined by a homogeneous linear polynomial $f = a_0z_0 + a_1z_1 + a_2z_2 \in \Bbbk[z_0, z_1, z_2]$ for some $a_0, a_1, a_2 \in \Bbbk$ not simultaneously zero.

(1) Show that two distinct points in $\mathbb{P}^2$ determine a unique line.

(2) Show that two distinct lines in $\mathbb{P}^2$ intersect at a unique point.

(*Hint:* How to compute the dimension of the null space of a matrix? Rank-nullity!)

Exercise 4.4. *Example of projective algebraic sets.* Recall that we always assume $\Bbbk$ is algebraically closed. Prove that projective algebraic sets in $\mathbb{P}^1$ are just the finite subsets in $\mathbb{P}^1$ (including $\varnothing$) together with $\mathbb{P}^1$ itself. You can follow these steps:

(1) Verify that they are indeed projective algebraic sets.

(2) Show that every non-constant homogeneous polynomial $f(z_0, z_1) \in \Bbbk[z_0, z_1]$ can be factored into a product of homogeneous polynomials of degree 1. (*Hint:* you can use the following lemma in algebra: a non-constant polynomial $g(x) \in \Bbbk[x]$ can be factored into a product of polynomials of degree 1.)

(3) Show that if a projective algebraic set in $\mathbb{P}^1$ is not $\mathbb{P}^1$ itself, then it contains at most finitely many points.