Exercise Sheet 5

This sheet will be discussed in the exercise class on 6 November. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 5.1. *Example: linear embedding and linear projection.*

(1) Show that $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3; [z_0 : z_1] \longmapsto [z_0 : z_1 : 0 : 0]$ is a morphism. Is it dominant? (*Remark:* in general, for any $n \leqslant m$, there is a *linear embedding* from $\mathbb{P}^n$ to $\mathbb{P}^m$ by identifying homogeneous coordinates in $\mathbb{P}^n$ with a subset of homogeneous coordinates in $\mathbb{P}^m$ and setting the remaining coordinates 0.)

(2) Show that $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1; [z_0 : z_1 : z_2 : z_3] \longmapsto [z_2 : z_3]$ is a rational map. Is it dominant? (*Remark:* in general, for any $m \geqslant n$, there is a *linear projection* from $\mathbb{P}^m$ to $\mathbb{P}^n$ by choosing a subset of the homogeneous coordinates in $\mathbb{P}^m$.)

(3) Is the composition $\psi \circ \varphi$ a well-defined rational map? Explain your reason.

Exercise 5.2. *Example: the cooling tower.* Consider $Y = \mathbb{V}(y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3$.

(1) Show that $y_0y_3 - y_1y_2$ is irreducible. Conclude that Y is a projective variety.

(2) Show that $\varphi : \mathbb{P}^2 \dashrightarrow Y; [x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_0x_1 : x_0x_2 : x_1x_2]$ is a rational map. Show that φ is dominant. (*Hint:* first show that each point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ with $y_0 \neq 0$ is in the image of φ, then use Lemma 5.16.)

(3) Show that $\psi : Y \dashrightarrow \mathbb{P}^2; [y_0 : y_1 : y_2 : y_3] \longmapsto [y_0 : y_1 : y_2]$ is a rational map. Show that ψ is dominant. (*Hint:* first show that each point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ with $x_0 \neq 0$ is in the image of ψ, then use Lemma 5.16.)

(4) Show that φ and ψ are birational maps. Conclude that Y is rational.

Exercise 5.3. *Example: the projective twisted cubic.* Consider the projective variety $Y = \mathbb{V}(y_0y_2 - y_1^2, y_1y_3 - y_2^2, y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3$.

(1) Show that $\varphi : \mathbb{P}^1 \longrightarrow Y; [u : v] \longmapsto [u^3 : u^2v : uv^2 : v^3]$ is a morphism.

(2) Show that φ is an isomophism by finding the inverse morphism $\psi : Y \longrightarrow \mathbb{P}^1$ and computing their compositions. Conclude that Y is isomorphic to $\mathbb{P}^1$.

Exercise 5.4. *A famous example: blow-up at a point.* Consider the projective variety $Y = \mathbb{V}(y_0y_2 - y_1^2,\ y_0y_4 - y_1y_3,\ y_1y_4 - y_2y_3) \subseteq \mathbb{P}^4$.

(1) Show $\varphi : \mathbb{P}^2 \dashrightarrow Y; [x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2]$ is a rational map.

(2) Show that φ is a birational map by finding the inverse rational map $\psi : Y \dashrightarrow \mathbb{P}^2$ and computing their compositions. Conclude that Y is rational.

(3) Show that ψ can be chosen to be a morphism. Show that ψ is surjective on points. Find all points $q \in Y$, such that $\psi(q) = [0 : 0 : 1]$.