Exercise Sheet 5

This sheet will be discussed in the exercise class on 6 November. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 5.1. Example: linear embedding and linear projection.

(1) Show that \( \varphi: \mathbb{P}^1 \to \mathbb{P}^3; [z_0 : z_1] \mapsto [z_0 : z_1 : 0 : 0] \) is a morphism. Is it dominant? (Remark: in general, for any \( n \leq m \), there is a linear embedding from \( \mathbb{P}^n \) to \( \mathbb{P}^m \) by identifying homogeneous coordinates in \( \mathbb{P}^n \) with a subset of homogeneous coordinates in \( \mathbb{P}^m \) and setting the remaining coordinates 0.)

(2) Show that \( \psi: \mathbb{P}^3 \to \mathbb{P}^1; [z_0 : z_1 : z_2 : z_3] \mapsto [z_2 : z_3] \) is a rational map. Is it dominant? (Remark: in general, for any \( m \geq n \), there is a linear projection from \( \mathbb{P}^m \) to \( \mathbb{P}^n \) by choosing a subset of the homogeneous coordinates in \( \mathbb{P}^m \).)

(3) Is the composition \( \psi \circ \varphi \) a well-defined rational map? Explain your reason.

Exercise 5.2. Example: the cooling tower. Consider \( Y = \mathbb{V}(y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3 \).

(1) Show that \( y_0y_3 - y_1y_2 \) is irreducible. Conclude that \( Y \) is a projective variety.

(2) Show that \( \varphi: \mathbb{P}^2 \to Y; [x_0 : x_1 : x_2] \mapsto [x_0^3 : x_0x_1 : x_0x_2 : x_1x_2] \) is a rational map. Show that \( \varphi \) is dominant. (Hint: first show that each point \( q = [y_0 : y_1 : y_2 : y_3] \in Y \) with \( y_0 \neq 0 \) is in the image of \( \varphi \), then use Lemma 5.16.)

(3) Show that \( \psi: Y \to \mathbb{P}^2; [y_0 : y_1 : y_2 : y_3] \mapsto [y_0 : y_1 : y_2] \) is a rational map. Show that \( \psi \) is dominant. (Hint: first show that each point \( p = [x_0 : x_1 : x_2] \in \mathbb{P}^2 \) with \( x_0 \neq 0 \) is in the image of \( \psi \), then use Lemma 5.16.)

(4) Show that \( \varphi \) and \( \psi \) are birational maps. Conclude that \( Y \) is rational.

Exercise 5.3. Example: the projective twisted cubic. Consider the projective variety \( Y = \mathbb{V}(y_0y_2 - y_1^2, y_1y_3 - y_2^2, y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3 \).

(1) Show that \( \varphi: \mathbb{P}^1 \to Y; [u : v] \mapsto [u^3 : u^2v : uv^2 : v^3] \) is a morphism.

(2) Show that \( \varphi \) is an isomorphism by finding the inverse morphism \( \psi: Y \to \mathbb{P}^1 \) and computing their compositions. Conclude that \( Y \) is isomorphic to \( \mathbb{P}^1 \).

Exercise 5.4. A famous example: blow-up at a point. Consider the projective variety \( Y = \mathbb{V}(y_0y_2 - y_1^2, y_0y_4 - y_1y_3, y_1y_4 - y_2y_3) \subseteq \mathbb{P}^4 \).

(1) Show \( \varphi: \mathbb{P}^2 \to Y; [x_0 : x_1 : x_2] \mapsto [x_0^3 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2] \) is a rational map.

(2) Show that \( \varphi \) is a birational map by finding the inverse rational map \( \psi: Y \to \mathbb{P}^2 \) and computing their compositions. Conclude that \( Y \) is rational.

(3) Show that \( \psi \) can be chosen to be a morphism. Show that \( \psi \) is surjective on points. Find all points \( q \in Y \), such that \( \psi(q) = [0 : 0 : 1] \).