Exercise Sheet 6

This sheet will be discussed in the exercise class on 13 November. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 6.1. *Example: the cooling tower, revisited.* Consider the projective algebraic set $Y = \mathbb{V}(y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3$. We know by Exercise 5.2 (1) that Y is a projective variety.

(1) Write down all standard affine pieces of Y.

(2) Explain why its function field $\Bbbk(Y) \cong \Bbbk(x_1, x_2)$. (*Hint:* you can use the results in Exercise 5.2 and any results mentioned in lectures.)

Exercise 6.2. *Example: irreducible cubic curves.*

(1) Show that the affine algebraic set $X = \mathbb{V}_a(y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)) \subseteq \mathbb{A}^2$ is an affine variety for any $\lambda_1, \lambda_2, \lambda_3 \in \Bbbk$.

(2) Find the projective closure $\overline{X} \subseteq \mathbb{P}^2$ of X and the points at infinity. Use Proposition 6.12 to conclude that $\overline{X}$ is a projective variety.

Exercise 6.3. *A caution for the projective closure.* We demonstrate Remark 6.9.

(1) Let $X = \mathbb{V}_a(I) \subseteq \mathbb{A}^3$ for the ideal $I = (f_1, f_2)$ in $\Bbbk[x, y, z]$ where $f_1 = y - x^2$ and $f_2 = z - x^3$. Using w as the extra variable, find polynomials $\overline{f_1}$ and $\overline{f_2}$ in $\Bbbk[w, x, y, z]$ which are the homogenisations of f_1 and f_2 respectively.

(2) We have seen in Exercise 2.4 (3) that $I = \mathbb{I}_a(X)$. Let $\overline{I}$ be the homogeneous ideal in $\Bbbk[w, x, y, z]$ defined as in Definition 6.5. Show that $y^2 - xz \in \overline{I}$ but $y^2 - xz \notin (\overline{f_1}, \overline{f_2})$. Conclude that $\overline{I} \neq (\overline{f_1}, \overline{f_2})$. Show that $\overline{X} \neq \mathbb{V}_p(\overline{f_1}, \overline{f_2})$.

Remark: this example demonstrates that the projective closure of an affine algebraic set X is not obtained simply by homogenising the generators of $\mathbb{I}_a(X)$ in general.

Exercise 6.4. *Geometric interpretation of the projective closure.* We consider $\mathbb{A}^n$ as the standard affine chart $U_0 \subseteq \mathbb{P}^n$. Then an affine algebraic set $X \subseteq \mathbb{A}^n$ can be thought as a subset of $\mathbb{P}^n$. Prove that its projective closure $\overline{X}$ is the smallest projective algebraic set in $\mathbb{P}^n$ containing X. You can follow these steps:

(1) Let $W \subseteq \mathbb{P}^n$ be any projective algebraic set that contains X. Let $g(z_0, z_1, \cdots, z_n) \in \mathbb{I}_p(W)$ be a homogeneous polynomial and $f(z_1, \cdots, z_n) = g(1, z_1, \cdots, z_n)$ the de-homogenisation of g. Show that $f \in \mathbb{I}_a(X)$.

(2) Let $\overline{f}$ be the homogenisation of f. Show that $g = z_0^k \cdot \overline{f}$ for some non-negative integer k. Conclude that $g \in \overline{I}$ where $\overline{I}$ is the homogenisation of the ideal $\mathbb{I}_a(X)$ defined as in Definition 6.5. Conclude that $\overline{X} \subseteq W$.

(3) Conclude that $\overline{X}$ is the smallest projective algebraic set in $\mathbb{P}^n$ containing X.