Exercise Sheet 7

This sheet will be discussed in the exercise class on 20 November. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 7.1. *Examples of affine varieties.* Find all singular points on the affine variety X, if there is any. In parts (1) – (3), you can assume the polynomial f is irreducible. In part (4), we know the two given polynomials generate $\mathbb{I}_a(X)$ by Exercise 2.4.

(1) $X = \mathbb{V}(f) \subseteq \mathbb{A}^2$ for $f = (x^2 + y^2)^3 - 4x^2y^2 \in \Bbbk[x, y]$.

(2) $X = \mathbb{V}(f) \subseteq \mathbb{A}^3$ for $f = xy^2 - z^2 \in \Bbbk[x, y, z]$.

(3) $X = \mathbb{V}(f) \subseteq \mathbb{A}^3$ for $f = xy + x^3 + y^3 \in \Bbbk[x, y, z]$.

(4) $X = \mathbb{V}(f, g) \subseteq \mathbb{A}^3$ for $f = y - x^2 \in \Bbbk[x, y, z]$ and $g = z - x^3 \in \Bbbk[x, y, z]$.

Exercise 7.2. *Example of projective varieties.* Show that the projective variety $X = \mathbb{V}(f) \subseteq \mathbb{P}^2$ for $f = xy - z^2 \in \Bbbk[x, y, z]$ is non-singular. Although one can achieve this by showing all three standard affine pieces are non-singular, it is not necessary to check every individual piece. Follow these steps for an easier approach.

(1) Show that the standard affine piece $X_0 = X \cap U_0$ is non-singular.

(2) Find out all points in $X \backslash X_0$. For each point $p \in X \backslash X_0$, use a standard affine piece of X that contains p to show X is non-singular at p.

(3) Using this method to find all singular points on the projective variety $\mathbb{V}(f) \subseteq \mathbb{P}^2$ for $f = x^3z + x^2yz + y^3z + x^4 + y^4$. You do not need to prove the irreducibility of any polynomial in this problem – just assume they are.

Exercise 7.3. *Example: plane cubics.* Find all singular points on the projective variety $\mathbb{V}(f) \subseteq \mathbb{P}^2$ where $f = y^2z - (x - \lambda_1 z)(x - \lambda_2 z)(x - \lambda_3 z)$ for some $\lambda_1, \lambda_2, \lambda_3 \in \Bbbk$, if there is any. You do not need to prove irreducibility of any polynomial in this problem.

(1) λ_1, λ_2 and λ_3 are distinct.

(2) $\lambda_1 = \lambda_2 \neq \lambda_3$.

(3) $\lambda_1 = \lambda_2 = \lambda_3$.

Exercise 7.4. *Example: projective twisted cubic.* Consider the projective variety $Y = \mathbb{V}_p(y_0y_2 - y_1^2, y_1y_3 - y_2^2, y_0y_3 - y_1y_2) \subseteq \mathbb{P}^3$. Follow the method in Example 7.23 to

(1) Determine whether Y is non-singular or singular.

(2) Compute the dimension of Y.

Remark: For any standard affine piece Y_i of Y, you can assume without proof that the dehomogenisation of the above three polynomials generate $\mathbb{I}_a(Y_i)$.