Exercise Sheet 8

This sheet will be discussed in the exercise class on 27 November. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 8.1. *Examples of rational curves.* Complete proofs of Propositions 8.7 and 8.18.

(1) Show that $L = \mathbb{V}(z) \subseteq \mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Conclude that L is rational.

(2) Show that φ_2 and ψ_2 defined in the proof of Proposition 8.18 are rational maps. Show that they are mutually inverse to each other. Conclude that the cuspidal cubic curve $C_2 = \mathbb{V}(y^2z - x^3) \subseteq \mathbb{P}^2$ is rational.

Exercise 8.2. *Example: Fermat cubic.* Consider the cubic curve $C = \mathbb{V}(x^3+y^3+z^3) \subseteq \mathbb{P}^2$.

(1) Show that C is non-singular.

(2) Show that the line $L = \mathbb{V}(z)$ meets C at 3 distinct points. Find all of them.

(3) For any $p = [a : b : c] \in C$, show that the tangent line $T_pC = \mathbb{V}(a^2x + b^2y + c^2z)$.

(4) Show that every point you find in part (2) is an inflection point.

Exercise 8.3. *Bézout's theorem for conics.* Prove Theorem 8.12 in these steps.

(1) If the conic $C = L_1 \cup L_2$ is the union of two lines, use Theorem 8.8 to conclude that $C \cap D$ comprises at most $2d$ distinct points; or precisely $2d$ points when multiplicities are counted. (*Remark:* if $L_1 \cap D$ and $L_2 \cap D$ have a common point p, the multiplicity at p is defined to be the sum of the two multiplicities at p.)

(2) If the conic C is irreducible, without loss of generality, we can assume $C = \mathbb{V}(xz - y^2)$ by Lemma 8.6. We have proved in Example 5.23 that every point in C can be given by $[p^2 : pq : q^2]$ for some $[p : q] \in \mathbb{P}^1$. Use the method in the proof of Theorem 8.8 to finish the proof.

Exercise 8.4. *An interesting application of Bézout's theorem.* Let $p_1, \cdots, p_5 \in \mathbb{P}^2$ be distinct points, and assume that no 4 of them are on the same line. Prove that there exists exactly one conic through all 5 points. You can follow these steps.

(1) Show that there exists at least one conic through all 5 points. (*Hint:* rank-nullity.)

(2) Suppose there are two distinct conics C_1 and C_2 through all 5 points. Use Bézout's theorem to conclude that they have a common component.

(3) If one of them is an irreducible conic, which has only one component, then the other must be the same irreducible conic, otherwise they cannot have a common component. Therefore both conics must be unions of two lines. Explain why we can assume $C_1 = L_0 \cup L_1$ and $C_2 = L_0 \cup L_2$ for distinct lines L_0, L_1 and L_2. Explain why this leads to a contradiction.