Exercise Sheet 9

This sheet will be discussed in the exercise class on 4 December. You are welcome to submit your solutions at the end of the exercise class or anytime earlier.

Exercise 9.1. *Example: understanding the simplified group law.*

(1) Show that $[0:1:0]$ is an inflection point of C in the simplified group law 9.4.

(2) In the simplified group law 9.4, explain briefly how to find all points $P \in C$ such that $P + P = O$.

(3) Consider the curve and the group law in Example 9.6. Let $A = [2:1:1]$ and $B = [-2:-1:1]$. Use the simplified group law to find out $-A$, $-B$ and $A + B$.

Exercise 9.2. *Example of group law.* Consider the non-singular cubic curve $C = \mathbb{V}(y^2z - x^3 - 4xz^2) \subseteq \mathbb{P}^2$. Let $O = [0:1:0]$ be the identity element in the group law.

(1) Find all points where C meets the line $L_1 = \mathbb{V}(z)$ and specify their multiplicities. Do the same for the lines $L_2 = \mathbb{V}(x)$ and $L_3 = \mathbb{V}(y - 2x)$.

(2) Find the order of the subgroup generated by the point $P = [2:4:1] \in C$.

(3) Find all points $Q \in C$ such that $Q + Q = O$.

Exercise 9.3. *Example: Tate's normal form.* Consider the projective closure C of the cubic curve $C_2 = \mathbb{V}(y^2 + sxy - ty - x^3 + tx^2) \subseteq \mathbb{A}^2$ for some fixed $s, t \in \Bbbk$ where $t \neq 0$. Assume C is non-singular. Let the point at infinity $O = [0:1:0]$ be the identity element in the group law on C.

(1) For any point $P = (a, b) \in C_2$, show that $-P = (a, -b - sa + t)$ in the group law.

(2) Suppose $Q = (0, 0) \in C_2$. Show that $Q + Q = (t, t(1 - s))$ in the group law.

Exercise 9.4. *Pascal's mystic hexagon.* Let $X \subseteq \mathbb{P}^2$ be an irreducible conic. Let $ABCDEF$ be a hexagon whose vertices are inscribed in X. Assume the three pairs of opposite sides meet in points P, Q, R respectively. (To be precise, the lines FA and CD meet at P; the lines AB and DE meet at Q; the lines BC and EF meet at R.) Show that P, Q, R are colinear. (That means, the three points are on the same line in $\mathbb{P}^2$.) You can follow these steps (the idea is already used in the proof of Proposition 9.13):

(1) Sketch a picture to illustrate the given situation.

(2) The three lines FA, BC and DE form a cubic curve C_1; the three lines AB, CD and EF form a cubic curve C_2. Find $C_1 \cap C_2$.

(3) Consider a third cubic C_3 given by the union of the conic X and the line PQ. Then apply Lemma 9.12.