Exercise Sheet 10

This sheet will be discussed in the exercise class on 7 December. You do not need to submit your solutions.

Exercise 10.1. *Infinitely many lines on planes.*

(1) Without loss of generality, we consider the plane $P = \mathbb{V}(z_0) \subseteq \mathbb{P}^3$. For every $[a : b : c] \in \mathbb{P}^2$, show that $\mathbb{V}(z_0, az_1 + bz_2 + cz_3)$ defines a line in P.

(2) Show that two such lines always meet at exactly one point.

Exercise 10.2. *Infinitely many lines on non-singular quadric surfaces.*

(1) Without loss of generality, we consider the quadric surface $Q = \mathbb{V}(z_0z_3 - z_1z_2) \subseteq \mathbb{P}^3$. Show that for every $[a : b] \in \mathbb{P}^1$, $\mathbb{V}(az_0 + bz_1, az_2 + bz_3)$ defines a line in Q.

(2) Show that two such lines are always disjoint.

(3) Show that every point in Q lies on exactly one of such lines.

(4) Can you write down another family of pairwisely disjoint lines in Q, such that every point in Q lies on exactly one of them?

Remark: the family of lines constructed in part (1) (or part (4)) is called a *ruling* on Q. We have seen in Exercise 5.2 that Q is birational to $\mathbb{P}^2$. This exercise shows that Q is not isomorphic to $\mathbb{P}^2$. The reason is: two lines in the same ruling on Q do not meet, while any two curves on $\mathbb{P}^2$ meet by Bézout's theorem. Since Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, it follows that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Exercise 10.3. *Rationality of a cubic surface.* Finish the proof of Proposition 10.11.

(1) Show that any point in the image of ψ satisfies the defining equation of S.

(2) Show that $\psi \circ \varphi$ and $\varphi \circ \psi$ are both identity maps on the loci where they are well-defined.

Remark: there is a general method to find out the explicit formula for a birational map between any given non-singular cubic surface S and $\mathbb{P}^2$. For that purpose we need to know the explicit equations of two disjoint lines on S. We do not discuss the details. However, the formula is usually very messy. The example in Proposition 10.11 is one of the very rare good-looking ones.

Exercise 10.4. *Thank you and have a wonderful Christmas vacation!*

Thank you all for your participation in this course. Please complete the Unit Evaluation for this course whenever convenient. If you have any questions during your revision, please feel free to ask me. There will be extra office hours after the vacation. You are also welcome to contact me by email at any time. Good luck with your exams!