EXTRA HINTS FOR EXERCISE SHEET 2

Exercise 2.1. For (1), just notice that each term in the binomial expansion has a factor a^m or b^n. (2) is straightforward. For (3), the first statement is an immediate consequence of Proposition 2.12 (1), and the second statement has to be checked by hand using definitions.

Exercise 2.2. For (1), both directions can be checked using definitions. For (2), you need to check $\sqrt{(f)} \supseteq (\overline{f})$ and $\sqrt{(f)} \subseteq (\overline{f})$. For "$\supseteq$", given any $g \in (\overline{f})$, let $m = \max\{k_1, \cdots, k_t\}$. You can show that $g^m \in (f)$. For "$\subseteq$", given any $g \in \sqrt{(f)}$, you need to show that g can be divided by each f_i. (3) is an immediate consequence of (2).

Exercise 2.3. For (1), you first need to show that every polynomial $f(x_1, \cdots, x_n) \in \mathbb{k}[x_1, \cdots, x_n]$ can be written in the form

$$f = (x_1 - a_1)g_1 + \cdots + (x_n - a_n)g_n + c$$

for some polynomials $g_1, \cdots, g_n \in \mathbb{k}[x_1, \cdots, x_n]$ and a constant $c \in \mathbb{k}$. There are two methods to prove this and you can use either one. You can look at Example 2 in the algebra review session for an argument similar to this. Once this formula is established, you can easily find $\ker \varphi_p$. To show m_p is a maximal ideal, you need to show that φ_p is surjective. Then the fundamental isomorphism theorem gives

$$\mathbb{k} \cong \mathbb{k}[x_1, \cdots, x_n]/m_p.$$

Since $\mathbb{k}$ is a field, Proposition 2.12 (1) implies m_p is a maximal ideal. Example 2 in the algebra review session is again a good model. For (2), $\mathbb{V}(m_p)$ is clearly a point. Proposition 2.16 shows that maximal ideals are one-to-one correspondent to points. For every point p, there is a maximal ideal m_p corresponding to it. Since points have been exhausted, the m_p's also exhaust all maximal ideals.

Exercise 2.4. For (1), you need to show $X \subseteq \mathbb{V}(I)$ and $X \supseteq \mathbb{V}(I)$, both of which are straightforward. For (2), you need to consider the ring homomorphism

$$\varphi : \mathbb{k}[x, y, z] \longrightarrow \mathbb{k}[t]; \quad f(x, y, z) \longmapsto f(t, t^2, t^3)$$

and use the fundamental isomorphism theorem. It is easy to find that φ is surjective, hence $\operatorname{im} \varphi = \mathbb{k}[t]$ is clear. To find out $\ker \varphi$, you need to first prove that every $f(x, y, z) \in \mathbb{k}[x, y, z]$ can be written in the form

$$f = (y - x^2)g_1 + (z - x^3)g_2 + h$$

where $g_1, g_2 \in \mathbb{k}[x, y, z]$ and $h \in \mathbb{k}[x]$. Once this is established, it is easy to find that $\ker \varphi = I$. The fundamental isomorphism theorem gives the required formula. The entire argument is similar to that of Example 2 in the algebra review session. Every step in (3) is straightforward.

Date: October 6, 2015.