EXTRA HINTS FOR EXERCISE SHEET 4

Exercise 4.1. In part (1), to see why $\mathbb{P}^1_{\mathbb{C}}$ is a bubble (or the surface of a globe), imagine the north pole is the extra point (the point in H_0). After removing this point, we have to identify the rest of the sphere with $\mathbb{A}^1 = \mathbb{C}$. This usually goes under the name "stereographic projection". You can find lots of nice pictures if you search for it in the Internet. In part (2), all you need to do is to substitute x_1 with $\frac{z_1}{z_0}$ and x_2 with $\frac{z_2}{z_0}$. After clearing denominators, set $z_0 = 0$ to find all points at infinity. Do not forget to write the points in homogeneous coordinates.

Exercise 4.2 (1). You can write the homogeneous decompositions of g and h as

$$g = g_M + g_{M-1} + \cdots + g_{m+1} + g_m,$$
$$h = h_N + h_{N-1} + \cdots + h_{n+1} + h_n,$$

where M and m are the maximal and minimal degrees of non-zero monomials in g respectively; similar for N and n. Then you need to show that f has non-zero terms of degree $M+N$ and $m+n$. As f is homogeneous, we must have $M+N = m+n$, which forces $M = m$ and $N = n$.

Exercise 4.2 (2). For one direction, assume I is a homogeneous ideal. Since $\Bbbk[z_0, \cdots, z_n]$ is a Noetherian ring, I is finitely generated. So we can write $I = (f_1, \cdots, f_m)$ for some $f_1, \cdots, f_m \in I$ which are not necessarily homogeneous polynomials. However, each f_i has a homogeneous decomposition, say, $f_i = f_{i,0} + f_{i,1} + \cdots + f_{i,d_i}$ where d_i is the degree of f_i. You can show that I is generated by all the $f_{i,j}$'s; that is,

$$I = (f_{1,0}, \cdots, f_{1,d_1}, f_{2,0}, \cdots, f_{2,d_2}, \cdots\cdots, f_{m,0}, \cdots f_{m,d_m}).$$

For the other direction, assume $I = (p_1, \cdots, p_l)$ for finitely many homogeneous polynomials $p_1, \cdots, p_l \in \Bbbk[z_0, \cdots, z_n]$, with $\deg p_i = e_i$. Given any polynomial $q \in I$, assume the homogeneous decomposition of q is $q = q_0 + \cdots + q_k$, where $k = \deg q$. To show I is a homogeneous ideal, we need to show that every $q_j \in I$. Since $q \in I$, we can write $q = p_1 r_1 + \cdots + p_l r_l$ for some $r_1, \cdots, r_l \in \Bbbk[z_0, \cdots, z_n]$. You can compare degree j terms on both sides to write each q_j in a similar expression, from which you can conclude $q_j \in I$.

Exercise 4.2 (3). Very similar to the proof in the affine case. See Lemma 1.10.

Exercise 4.3. For part (1), let the two points be $p = [p_0 : p_1 : p_2]$ and $q = [q_0 : q_1 : q_2]$. If a line $\mathbb{V}(a_0 z_0 + a_1 z_1 + a_2 z_2)$ passes through the two points, you can find out that the coefficients a_0, a_1, a_2 have to satisfy two linear homogeneous equations. Use the theorem

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of rank-nullity to show that the null space has dimension 1. Explain why that means the line is unique. Part (2) is very similar.

Exercise 4.4. Very similar to the affine case. See Exercise 1.3. For part (1), do not forget that you need homogeneous polynomials. For part (2), let $f \in \Bbbk[z_0, z_1]$ be a homogeneous polynomial of degree d. Assume z_0^e be the highest power of z_0 dividing f for some $e \leqslant d$. Then we can write

$$\begin{aligned} f &= c_0 z_0^d + c_1 z_0^{d-1} z_1 + \cdots + c_{d-e} z_0^e z_1^{d-e} \\ &= z_0^d \cdot \left(c_0 + c_1 \frac{z_1}{z_0} + \cdots + c_{d-e} \frac{z_1^{d-e}}{z_0^{d-e}} \right). \end{aligned}$$

We consider the polynomial $g(x) = c_0 + c_1 x + \cdots + c_{d-e} x^{d-e}$ and apply the lemma in the hint. After substituting x with $\frac{z_1}{z_0}$, notice that the factor z_0^d can be used to cancel all denominators. For part (3), assume a projective algebraic set $X \subseteq \mathbb{P}^1$ is given by $X = \mathbb{V}(S)$ for a finite set S of homogeneous polynomials in $\Bbbk[z_0, z_1]$. If S does not have any non-zero polynomial then $X = \mathbb{P}^1$. Otherwise, assume $f \in S$ is a non-zero homogeneous polynomial of degree d. Use part (2) to show that X comprises at most d points.