EXTRA HINTS FOR EXERCISE SHEET 5

Exercise 5.1. You can follow the method in the examples given in lectures to check φ is a morphism and ψ is a rational map. To show φ is not dominant, you can use Definition 5.14 with $W = \mathbb{V}(z_2, z_3)$ (there are also many other possible choices for W). To show ψ is dominant, you can use Lemma 5.16 with $Z = \varnothing$. The composition $\psi \circ \varphi$ is nowhere defined (you need to explain the reason) hence violates the second requirement in the definition of a rational map.

Exercise 5.2. To show $y_0y_3 - y_1y_2$ is irreducible, you can think of it as a degree 1 polynomial in y_0. If it can be written as the product of two polynomials, say $f \cdot g$, then the two factors have degrees 1 and 0 in y_0 respectively. Without loss of generality, we can write $f = f_1 \cdot y_0 + f_0$ and $g = g_0$, where f_1, f_0 and g_0 are polynomials in y_1, y_2 and y_3 (i.e. y_0 does not occur). By comparing the coefficients of y_0 and terms without y_0, you can find that g_0 must be a constant.

To show φ and ψ are rational maps, follow the examples given in lectures. To show φ and ψ are dominant, follow the hint in each part and use Lemma 5.16. More explicitly, in part (2), you can check that a point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ with $y_0 \neq 0$ is the image of $p = [y_0 : y_1 : y_2] \in \mathbb{P}^2$; in part (3), you can check that a point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ with $x_0 \neq 0$ is the image of $q = [x_0 : x_1 : x_2 : \frac{x_1x_2}{x_0}] \in Y$. To show φ and ψ are birational, compute both compositions.

Exercise 5.3. This exercise is very similar to Examples 5.11, 5.12 and 5.23. The inverse map $\psi : Y \longrightarrow \mathbb{P}^1$ is given by $\psi([y_0 : y_1 : y_2 : y_3]) = [y_0 : y_1]$ or $[y_2 : y_3]$. There are details to check, which are similar to the examples given in lectures.

Exercise 5.4. This exercise is still quite similar to the previous one, but there are more details to check. The inverse map $\psi : Y \longrightarrow \mathbb{P}^2$ required in part (2) can be defined by $\psi([y_0 : y_1 : y_2 : y_3 : y_4]) = [y_0 : y_1 : y_3]$ or $[y_1 : y_2 : y_4]$. For part (3), you can find that $[0 : 0 : 1] = \psi(q)$ for any $q \in Y$ in the form of $q = [0 : 0 : 0 : y_3 : y_4]$ where y_3 and y_4 are not simultaneously zero. In particular, $[0 : 0 : 1]$ is in the image of ψ. For any point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ such that $p \neq [0 : 0 : 1]$, then either $x_0 \neq 0$ or $x_1 \neq 0$. In both cases, you can check that $p = \psi([x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2])$ is still in the image of ψ.

Date: October 29, 2015.