EXTRA HINTS FOR EXERCISE SHEET 6

Exercise 6.1. Example 6.4 could be helpful for part (1). Example 6.18 and Proposition 6.21 could be helpful for part (2).

Exercise 6.2. Part (1) is very similar to Exercises 3.2 (3) and 3.3 (3). Examples 6.10 and 6.11 could be helpful for part (2). Unfortunately I didn't say too much about Proposition 6.12 in the lecture. But part of the content in Proposition 6.12 is that the projective closure of an affine variety is a projective variety, which is what we need here.

Exercise 6.3. It is probably more important to understand why this exercise is meaningful. Indeed, this exercise demonstrate a tricky point in the construction of the projective closure of an affine algebraic set X. We know from Proposition 6.8 that if X is defined by just one polynomial, then we can homogenise this polynomial to obtain $\overline{X}$. This exercise shows: if X is defined by two polynomials, then the homogenisation of the two polynomials do not always give the correct projective closure. This is precisely the reason why Definition 6.5 is somehow unintuitive. Indeed, Definition 6.5 tells us that we need to homogenise *all* polynomials in $\mathbb{I}(X)$ to get the correct projective closure of X.

This exercise is probably easier than it looks. Part (1) is similar to Example 6.4. There are three questions in part (2). To prove $y^2 - xz \in \overline{I}$, you need to show first that $y^2 - xz \in I$ by writing it in the form of $f_1 g_1 + f_2 g_2$ for some $g_1, g_2 \in \Bbbk[x, y, z]$. This might require some trials and errors. Then by the construction of $\overline{I}$ in Definition 6.5, the homogenisation of $y^2 - xz$ should be in $\overline{I}$, which is still $y^2 - xz$. To prove $y^2 - xz \notin (\overline{f_1}, \overline{f_2})$, you need to explain why it is impossible to write $y^2 - xz$ in the form of $\overline{f_1} h_1 + \overline{f_2} h_2$ for any $h_1, h_2 \in \Bbbk[w, x, y, z]$. This is fun and requires probably just one sentence if you look at it in the correct way and I don't want to be a spoiler here :-) Recall that $\overline{X} = \mathbb{V}_p(\overline{I})$ is the projective closure of X. To show $\overline{X} \neq \mathbb{V}_p(\overline{f_1}, \overline{f_2})$, you need to explain why the right-hand side contains some points which are not in the left-hand side. It suffices to find one such point. More precisely, you want this point to be a solution to $\overline{f_1}$ and $\overline{f_2}$, but fail to be a solution to some polynomial in $\overline{I}$, such as $y^2 - xz$. What could this point be?!

Exercise 6.4. For part (1), we need to show that $f(p) = 0$ for eery $p \in X$. The key is: if the non-homogeneous coordinates of p are given by $p = (a_1, \cdots, a_n)$ as a point in U_0, then the homogeneous coordinates of p can be given by $p = [1 : a_1 : \cdots : a_n]$; see Construction 4.5 in lecture notes. For part (2), assume that g is a homogeneous polynomial of degree d, and that z_0^k is the highest power of z_0 dividing g. Then we can collect terms containing

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the same power of z_0 and write g in the form of

$$g = z_0^k \cdot f_{d-k} + z_0^{k+1} \cdot f_{d-k-1} + \cdots + z_0^{d-1} \cdot f_1 + z_0^d \cdot f_0$$

where each f_i is a homogeneous polynomial of degree i, and z_0 does not occur in f_i for $i = 0, 1, \cdots, d-k$. Try to use it to work out a formula for f, and a formula for $\overline{f}$. Then you will see $g = z_0^k \cdot f$. Explain why $\overline{f} \in I$, which implies $g \in I$. Since every homogeneous polynomial in $\mathbb{I}_p(W)$ is a homogeneous polynomial in $\overline{I}$, we have $\mathbb{V}_p(\mathbb{I}_p(W)) \supseteq \mathbb{V}_p(\overline{I})$. That is, $W \supseteq \overline{X}$. Part (3) follows from a combination of parts (1) and (2).