EXTRA HINTS FOR EXERCISE SHEET 7

Exercise 7.1. Parts (1)–(3) are all similar to Example 7.3. Caution in part (2): there could be infinitely many singular points. Caution in part (3): we consider f as a polynomial in three indeterminants. Since z does not explicitly occur in f, $\frac{\partial f}{\partial z} = 0$. Part (4) requires Definition 7.13. The methord for the affine piece X_0 in Example 7.23 can be used for part (4).

Exercise 7.2. Parts (1) and (2) show how to find singular points in an irreducible projective hypersurface. Part (1) is very similar to Example 7.3. In part (2), you need to find all points $p \in X \backslash X_0$ first. Indeed, there is only one such point p. Then you need to determine which standard affine piece of X contains p, and determine whether p is singular in this standard affine piece. For this you only need to compute the relevant partial derivatives at the point p. p is a singular point if both partial derivatives vanish; otherwise p is a non-singular point.

Part (3) is another example of this calculation. You need to start with a certain standard affine piece, for example $X_0 = X \cap U_0$, by setting $x = 1$ in the defining polynomial f. If you look at the equations for the singular points in the correct way, it is easy to see that they have no common solution, hence X_0 is non-singular. It remains to deal with the points in $X \backslash X_0$. By setting $x = 0$ you can find all points in $X \backslash X_0$. There are two of them. From their homogeneous coordinates you can decide which standard affine piece contains them. Then you need to determine whether they are singular points in this standard affine piece.

Exercise 7.3. This exercise is very similar to Exercise 7.2. You need to first deal with one standard affine piece and then consider all remaining points. You can choose any standard affine piece to start with. But it might be easier to start with $X_2 = X \cap U_2$. You can obtain X_2 by setting $z = 1$ in f. Following the method in Example 7.3, you will find X_2 is non-singular in case (1) but has one singular point in cases (2) and (3). To find out all points in $X \backslash X_2$, we need to set $z = 0$ in f. Then we can find the only point $p = [0 : 1 : 0]$. Then we just need to check it is a non-singular point in the standard affine piece $X_1 = X \cap U_1$.

Exercise 7.4. This exercise is similar to Example 7.23. We start with the standard affine piece $Y_0 = Y \cap U_0$, which can be obtained by setting $y_0 = 1$ in the defining equations of Y. At each point $p \in Y_0$, the matrix of partial derivatives M_p is a 3×3 matrix in this case. Notice that its determinant is 0, and there are two linearly independent rows (or columns). Hence $\operatorname{rank} M_p = 2$ at every $p \in Y_0$, then $\dim T_p Y_0 = 1$. It follows that Y_0 is non-singular and $\dim Y_0 = 1$, which further implies $\dim Y = 1$. It remains to consider the points in $Y \backslash Y_0$. Assume $p = [y_0 : y_1 : y_2 : y_3]$ is such a point, then $y_0 = 0$. We can use the defining equations of Y to find that $p = [0 : 0 : 0 : 1]$. Then we can use the standard affine piece $Y_3 = Y \cap U_3$ to prove that it is a non-singular point.

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