EXTRA HINTS FOR EXERCISE SHEET 8

Exercise 8.1. For (1), you need to write down two morphisms and prove that they are mutually inverse. If you need more help, you can consider the linear embeddings and linear projections studied in Exercise 5.1. For (2), you can use the first part in the proof of Proposition 8.18 as a model.

Exercise 8.2. For (1), you just need to check one standard affine piece of C is non-singular, then use the symmetry in the defining polynomial of C to claim that the same calculation applies to all other standard affine pieces. For (2), you need to follow the steps in the proof of Theorem 8.8. Here a point $p \in L$ can be given by $p = [x : y : 0]$ since $z = 0$. (There is no harm to assume the field $\Bbbk$ is the field of complex numbers $\mathbb{C}$ if you need.) For (3), at least one of the three coordinates of the point p is non-zero. Without loss of generality, we can assume $a \neq 0$, then this point is in the standard affine piece $C_0 = C \cap U_0$ with non-homogeneous coordinates $p = (\frac{b}{a}, \frac{c}{a})$. We can use Definition 7.5 to write down the tangent space T_pC_0 of the standard affine piece C_0 at the point p, then use Definition 7.9 to take the projective closure of T_pC_0 to get T_pC. You can simplify the polynomial to get the required form. For part (4), for each point p you find in part (2), you can use part (3) to easily write down the polynomial that defines the tangent line T_pC. If you follow the proof of Theorem 8.8 to compute the intersection of T_pC and C, you can find that they meet at only one point with multiplicity 3. Finally Definition 8.20 tells you that the point p is an inflection point.

Exercise 8.3. There is not much to say for part (1). You just need to realise that a common point of C and D must be either a common point of L_1 and D or a common point of L_2 and D. For part (2), the proof is parallel to that of Theorem 8.8.

Exercise 8.4. For (1), the idea is that a conic is given by a homogeneous polynomial of degree 2, which has 6 independent coefficients, as explained in Example 8.3. Each point p_i gives a linear constraint on these coefficients, hence the 6 coefficients have to satisfy 5 linear constraints. The rank-nullity implies there are non-zero solutions. Such a solution determines a homogeneous polynomial $g(x, y, z)$ of degree 2, as in Example 8.3. But you need to explain why it does not have repeated factors. If it does, then g defines a double line, with all 5 points on it. This is a contradiction. Part (2) follows immediately from Theorem 8.12. For (3), the key point is that if a certain p_i is not on L_0, then it must be on both L_1 and L_2. But L_1 and L_2 have only one common point by Theorem 8.8. Why does it lead to a contradiction?

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