1.2. **Noetherian rings and Hilbert basis theorem.** We start with some algebra. But eventually we will see its geometric applications.

Recall that a ring R is a principal ideal domain (or PID) if every ideal of R is generated by one element. PIDs have many good properties. But unfortunately many interesting rings in algebraic geometry, for example, $\Bbbk[x_1, \cdots, x_n]$ when $n \geqslant 2$, are not PIDs. It will be helpful to generalise the notion of PID to include examples like these.

Definition 1.12. A ring R is *Noetherian* if every ideal of R is finitely generated.

It is immediately clear from the definition that every PID is Noetherian. We want to see more examples. A powerful tool to produce such examples is the following

Theorem 1.13 (Hilbert Basis Theorem)**.** *If a ring R is Noetherian, then $R[x]$ is also Noetherian.*

Proof. Non-examinable. Interested reader can find the proof in [Section 3.3, Reid, Undergraduate Algebraic Geometry] or [Section 1.4, Fulton, Algebraic Curves]. □

Corollary 1.14. *For any field $\Bbbk$ and $n \in \mathbb{Z}_+$, the ring $\Bbbk[x_1, \cdots, x_n]$ is Noetherian.*

Proof. We prove by induction on n. When $n = 1$, we know $\Bbbk[x_1]$ is a PID, hence is Noetherian. Assume $R_n = \Bbbk[x_1, \cdots, x_n]$ is a Noetherian ring. We need to show that $R_{n+1} = \Bbbk[x_1, \cdots, x_n, x_{n+1}]$ is also Noetherian. Notice that by collecting terms with respect to the variable x_{n+1}, every polynomial in R_{n+1} can be written as a polynomial in x_{n+1} with coefficients in R_n. In other words, we have $R_{n+1} = R_n[x_{n+1}]$. By Hilbert Basis Theorem 1.13 and the induction assumption, we conclude that R_{n+1} is Noetherian. □

There is yet another powerful tool very useful for producing examples of Noetherian rings. Before stating it we need to give an equivalent description of a Noetherian ring.

Proposition 1.15. *A ring R is Noetherian if and only if the following ascending chain condition (or ACC) holds: for every ascending chain of ideals in R*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots,$$

there exists a positive integer N such that $I_n = I_N$ for all $n \geqslant N$.

Proof. (This proof is non-examinable and not covered in lectures.)

We first prove that the Noetherian condition implies ACC. Take any ascending chain of ideals in R, say, $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$. Set $I = \cup_{n=1}^{\infty} I_n$. We claim that I is an ideal in R. Indeed, for any $r \in R$ and $a, b \in I$, assume $a \in I_i$ and $b \in I_j$. Then $a, b \in I_{\max\{i,j\}}$. It follows that $a + b \in I_{\max\{i,j\}}$, hence $a + b \in I$. Moreover, $ra \in I_i$ hence $ra \in I$. This concludes that I is an ideal.

Since R is Noetherian, I is finitely generated, say, $I = (f_1, \cdots, f_m)$. Then each f_i is an element in I_{n_i} for some n_i. Take $N = \max\{n_1, \cdots, n_m\}$. We claim that $I_N = I$. On one hand $f_i \in I_{n_i} \subseteq I_N$ for every i, hence $r_1 f_1 + \cdots + r_m f_m \in I_N$ for any $r_1, \cdots, r_m \in R$, which implies $I \subseteq I_N$. On the other hand we have $I_N \subseteq I$ by the construction of I. It follows that $I_N = I$. For every $n \geqslant N$, we have $I_N \subseteq I_n \subseteq I = I_N$, hence $I_n = I_N$.

We then prove that ACC implies the Noetherian condition. We use contradiction. Assume R has an ideal J which is not finitely generated. We pick an element $g_1 \in J$ and define $I_1 = (g_1)$. Since J is not finitely generated we have $I_1 \subsetneq J$, hence we can pick an element $g_2 \in J \backslash I_1$ and define $I_2 = (g_1, g_2)$. Similarly we can pick $g_3 \in J \backslash I_2$ and define $I_3 = (g_1, g_2, g_3)$. Repeat this process indefinitely, we get a chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ where each $I_i = (g_1, \cdots, g_i)$. Every inclusion in the chain is strict, hence the chain never stabilises, which is a contradiction to ACC. □

Now we are ready to state our second tool for producing examples of Noetherian rings.

Proposition 1.16. *Let R be a Noetherian ring and I is an ideal in R. Then the quotient ring R/I is also Noetherian.*

Proof. We leave the proof as an exercise. □

Corollary 1.17. *For any ideal I in $\Bbbk[x_1, \cdots, x_n]$, $\Bbbk[x_1, \cdots, x_n]/I$ is a Noetherian ring.*

Proof. This is a consequence of Corollary 1.14 and Proposition 1.16. □

Why are we so interested in Noetherian rings? Can we understand more geometry from the fact that $\Bbbk[x_1, \cdots, x_n]$ is Noetherian? The following is the answer.

Theorem 1.18. *Let $X \subseteq \mathbb{A}^n$ be an algebraic set, such that $\varnothing \neq X \neq \mathbb{A}^n$. Then X is the intersection of finitely many hypersurfaces.*

Proof. By Lemma 1.10, we can write $X = \mathbb{V}(I)$ for some ideal I in $\Bbbk[x_1, \cdots, x_n]$. By Corollary 1.14, I is finitely generated, say, $I = (f_1, \cdots, f_m)$. By Lemma 1.10 again we can write $X = \mathbb{V}(I) = \mathbb{V}(f_1, \cdots, f_m)$. Without loss of generality, we can assume every f_i is non-constant. Indeed, if a certain f_i is zero, then we can simply remove it from the set of generators; if a certain f_i is a non-zero constant, then $X = \varnothing$ which is excluded by the assumption. Notice that

$$\begin{aligned} X &= \mathbb{V}(f_1, \cdots, f_m) \\ &= \{p \in \mathbb{A}^n \mid f_1(p) = \cdots = f_m(p) = 0\} \\ &= \{p \in \mathbb{A}^n \mid f_1(p) = 0\} \cap \cdots \cap \{p \in \mathbb{A}^n \mid f_m(p) = 0\} \\ &= \mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_m). \end{aligned}$$

Since each $\mathbb{V}(f_i)$ is a hypersurface in $\mathbb{A}^n$, we conclude that X is the intersection of finitely many hypersurfaces. □

Equivalently, we can say that every algebraic set in $\mathbb{A}^n$ can be defined by finitely many polynomials (this even includes the algebraic sets $\varnothing$ and $\mathbb{A}^n$, as they are defined by $\{1\}$ and $\{0\}$ respectively). Notice that a geometric result like Theorem 1.18 cannot be obtained without the algebraic theory of Noetherian rings. In fact, thoroughout this course, we will always strive to build up a bridge, or a dictionary, between geometry and algebra. How to translate a geometric question into algebra, and how to interpret an algebraic result in the geometric language, will always be our main themes in this course.