2.2. Prime ideals and maximal ideals. We have established a one-to-one correspondence (2.2) between radical ideals in \( k[x_1, \ldots, x_n] \) and algebraic sets in \( \mathbb{A}^n \). A major benefit: we can read off some geometric properties of algebraic sets from algebraic properties of the corresponding radical ideals. We will see two such examples in this lecture.

**Definition 2.10.** Let \( I \) be an ideal in a ring \( R \).

1. The ideal \( I \) is prime if it is proper, and \( fg \in I \) implies \( f \in I \) or \( g \in I \).
2. The ideal \( I \) is maximal if it is proper, and for any ideal \( J \) satisfying \( I \subseteq J \subseteq R \), we have either \( J = I \) or \( J = R \).

**Example 2.11.** We look at some ideals in \( k[x] \).

1. Consider \( I_1 = (x^2 - x) \). \( I_1 \) is not prime because \( x(x - 1) \in I_1 \), while \( x \notin I_1 \) and \( x - 1 \notin I_1 \). \( I_1 \) is not maximal because \( (x^2 - x) \nsubseteq (x) \nsubseteq k[x] \).
2. Consider \( I_2 = (x) \). We claim \( (x) \) is prime. Assume \( fg \in (x) \), then \( fg = xh \) for some \( h \in k[x] \). By unique factorisation, since \( x \) is irreducible, it must be a factor of \( f \) or \( g \). Hence \( f \in (x) \) or \( g \in (x) \). We claim \( (x) \) is maximal. Assume \( (x) \subseteq J \subseteq k[x] \). If \( I \neq (x) \), then there exists \( f \in I \setminus (x) \). Write \( f = a_0 + a_1x + \cdots + a_nx^n \), then \( a_0 \neq 0 \), since otherwise \( f \in (x) \). We observe \( f - a_0 = a_1x + \cdots + a_nx^n \in (x) \subseteq I \). It follows that \( a_0 \in I \), hence \( I = k[x] \) since \( a_0 \) is a unit in \( k[x] \).
3. Consider \( I_3 = (0) \). \( I_3 \) is prime because \( fg = 0 \) implies that either \( f = 0 \) or \( g = 0 \) as \( k[x] \) is an integral domain. \( I_3 \) is not maximal because \( (0) \nsubseteq (x) \nsubseteq k[x, y] \).

**Proposition 2.12.** Let \( I \) be an ideal in the ring \( R \).

1. \( I \) is a prime ideal if and only if \( R/I \) is an integral domain. \( I \) is a maximal ideal if and only if \( R/I \) is a field.
2. Every maximal ideal is prime. Every prime ideal is radical.

**Proof.** (1) is non-examinable. (2) is an exercise. \( \square \)

Under the correspondence (2.2), we will find out what prime and maximal ideals correspond to. Now we switch to geometry.

**Definition 2.13.** An algebraic set \( X \subseteq \mathbb{A}^n \) is irreducible if there does not exist a decomposition of \( X \) as a union of two strictly smaller algebraic sets. An irreducible (affine) algebraic set is also called an affine variety. An algebraic set \( X \subseteq \mathbb{A}^n \) is reducible if it is not irreducible.

**Example 2.14.** We look at some algebraic sets in \( \mathbb{A}^2 \).
(1) The algebraic set \( \mathbb{V}(xy) \subseteq \mathbb{A}^2 \) is the union of two coordinate axes. In other words, \( \mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y) \). Since each coordinate axis is an algebraic set strictly smaller than \( \mathbb{V}(xy) \), we conclude that \( \mathbb{V}(xy) \) is reducible.

(2) The algebraic set \( \mathbb{V}(x,y) \subseteq \mathbb{A}^2 \) consists of just one point, hence there is no way to decompose it as the union of two strictly smaller algebraic sets. It follows that \( \mathbb{V}(x,y) \) is irreducible. Similarly, a point is always irreducible.

Next we show that prime ideals correspond to irreducible algebraic sets.

**Proposition 2.15.** Let \( I \) be a radical ideal in \( k[x_1, \ldots, x_n] \) and \( X = \mathbb{V}(I) \) the algebraic set in \( \mathbb{A}^n \) defined by \( I \). Then \( I \) is prime if and only if \( X \) is irreducible.

**Proof.** In fact we prove the contrapositive: \( X \) is reducible \( \iff \) \( I \) is not prime.

We first prove \( \implies \). Suppose \( X = X_1 \cup X_2 \) with algebraic sets \( X_1, X_2 \subseteq X \). Then \( X_1 \subseteq X \) implies that \( \mathbb{I}(X_1) \supseteq \mathbb{I}(X) \) by Proposition 2.9 (2). Hence there exists \( f_1 \in \mathbb{I}(X_1) \setminus \mathbb{I}(X) \).

Similarly \( X_2 \subseteq X \) implies that there exists \( f_2 \in \mathbb{I}(X_2) \setminus \mathbb{I}(X) \). The product \( f_1 f_2 \) vanishes at all points of \( X \), hence \( f_1 f_2 \in \mathbb{I}(X) \). Therefore \( I = \mathbb{I}(X) \) is not prime.

We then prove \( \Leftarrow \). Since \( I \) is not prime, there exist \( f_1, f_2 \notin I \) such that \( f_1 f_2 \in I \). Consider the set \( S_1 = I \cup \{ f_1 \} \). Then \( X_1 = \mathbb{V}(S_1) \) is an algebraic set. Since \( S_1 \supseteq I \), we have \( X_1 \subseteq X \) by Proposition 1.7. Moreover, since \( f_1 \notin I \), there is some point \( p \in X \) such that \( f_1(p) \neq 0 \), therefore \( p \notin X_1 \). It follows that \( X_1 \subseteq X \). Similarly we can consider \( S_2 = I \cup \{ f_2 \} \), then \( X_2 = \mathbb{V}(S_2) \subseteq X \).

It remains to show that \( X_1 \cup X_2 = X \). Since \( X_1 \) and \( X_2 \) are subsets of \( X \), we have \( X_1 \cup X_2 \subseteq X \). Conversely, for any \( p \in X \), \( f(p) = 0 \) for every \( f \in I \). Moreover \( f_1(p) f_2(p) = 0 \), which implies \( f_1(p) = 0 \) or \( f_2(p) = 0 \). Therefore \( p \in \mathbb{V}(S_1) = X_1 \) or \( p \in \mathbb{V}(S_2) = X_2 \). This implies \( X \subseteq X_1 \cup X_2 \).

Finally we show that maximal ideals correspond to points.

**Proposition 2.16.** Let \( I \) be a radical ideal in \( k[x_1, \cdots, x_n] \) and \( X = \mathbb{V}(I) \) the algebraic set in \( \mathbb{A}^n \) defined by \( I \). Then \( I \) is maximal if and only if \( X \) is a point.

**Proof.** (This proof is non-examinable and not covered in lectures.)

In fact we prove the contrapositive: \( X \) is not a point \( \iff \) \( I \) is not maximal.

We first prove \( \implies \). If \( X \) is not a point, then either \( X = \emptyset \) or \( X \) contains more than one point. If \( X = \emptyset \), then by Proposition 2.9 (1), \( I = \mathbb{I}(X) = k[x_1, \cdots, x_n] \) is not a proper ideal hence not maximal. If \( X \) contains more than one point, then we can pick a subset \( Y \) of \( X \) containing only one point. Hence we have \( \emptyset \subseteq Y \subseteq X \). By Proposition 2.9 (2), we have \( k[x_1, \cdots, x_n] = \mathbb{I}(\emptyset) \supseteq \mathbb{I}(Y) \supseteq \mathbb{I}(X) \). Hence \( I = \mathbb{I}(X) \) is not maximal.
We then prove “⇐”. If \( I \) is not maximal, then either \( I \) is not a proper ideal, or there exists an ideal \( J \) such that \( I \subseteq J \subsetneq \mathbb{k}[x_1, \ldots, x_n] \). If \( I \) is not proper then \( I = \mathbb{k}[x_1, \ldots, x_n] \), hence \( X = \mathcal{V}(I) = \emptyset \) which is not a point. If \( I \subsetneq J \subsetneq \mathbb{k}[x_1, \ldots, x_n] \) for some ideal \( J \), then we claim that we actually have \( I \subsetneq \sqrt{J} \subsetneq \mathbb{k}[x_1, \ldots, x_n] \). Indeed, by Lemma 2.2, we have \( I \subsetneq J \subsetneq \sqrt{J} \). Moreover, by Nullstellensatz 2.8 (1), we have \( \mathcal{V}(J) \neq \emptyset \), hence \( \sqrt{J} = \mathbb{I}((\mathcal{V}(J)) \subsetneq \mathbb{k}[x_1, \ldots, x_n] \). Armed with this claim we apply Proposition 2.9 (2) to get \( \mathcal{V}(I) \supseteq \mathcal{V}(\sqrt{J}) \supseteq \emptyset \). It follows that \( \mathcal{V}(\sqrt{J}) \) contains at least one point, hence \( X = \mathcal{V}(I) \) contains more than one point. \( \square \)

In summary, the \( \mathcal{V} - \mathbb{I} \) correspondences induce bijections in each row of the diagram:

\[
\begin{array}{ccc}
\{\text{radical ideals in } \mathbb{k}[x_1, \ldots, x_n]\} & \overset{\mathcal{V}}{\longrightarrow} & \{\text{algebraic sets in } \mathbb{A}^n\} \\
\downarrow & & \downarrow \\
\{\text{prime ideals in } \mathbb{k}[x_1, \ldots, x_n]\} & \overset{\mathbb{I}}{\longrightarrow} & \{\text{irreducible algebraic sets in } \mathbb{A}^n\} \\
\downarrow & & \downarrow \\
\{\text{maximal ideals in } \mathbb{k}[x_1, \ldots, x_n]\} & \overset{\mathcal{V}}{\longrightarrow} & \{\text{points in } \mathbb{A}^n\} \\
\end{array}
\]