2.2. **Prime ideals and maximal ideals.** We have established a one-to-one corrspondence (2.2) between radical ideals in $\Bbbk[x_1, \cdots, x_n]$ and algebraic sets in $\mathbb{A}^n$. A major benefit: we can read off some geometric properties of algebraic sets from algebraic properties of the corresponding radical ideals. We will see two such examples in this lecture.

Definition 2.10. Let I be an ideal in a ring R.

(1) The ideal I is *prime* if it is proper, and $fg \in I$ implies $f \in I$ or $g \in I$.

(2) The ideal I is *maximal* if it is proper, and for any ideal J satisfying $I \subseteq J \subseteq R$, we have either $J = I$ or $J = R$.

Example 2.11. We look at some ideals in $\Bbbk[x]$.

(1) Consider $I_1 = (x^2 - x)$. I_1 is not prime because $x(x-1) \in I_1$, while $x \notin I_1$ and $x - 1 \notin I_1$. I_1 is not maximal because $(x^2 - x) \subsetneq (x) \subsetneq \Bbbk[x]$.

(2) Consider $I_2 = (x)$. We claim (x) is prime. Assume $fg \in (x)$, then $fg = xh$ for some $h \in \Bbbk[x]$. By unique factorisation, since x is irreducible, it must be a factor of f or g. Hence $f \in (x)$ or $g \in (x)$. We claim (x) is maximal. Assume $(x) \subseteq I \subseteq \Bbbk[x]$. If $I \neq (x)$, then there exists $f \in I \backslash (x)$. Write $f = a_0 + a_1 x + \cdots + a_n x^n$, then $a_0 \neq 0$, since otherwise $f \in (x)$. We observe $f - a_0 = a_1 x + \cdots + a_n x^n \in (x) \subseteq I$. It follows that $a_0 \in I$, hence $I = \Bbbk[x]$ since a_0 is a unit in $\Bbbk[x]$.

(3) Consider $I_3 = (0)$. I_3 is prime because $fg = 0$ implies that either $f = 0$ or $g = 0$ as $\Bbbk[x]$ is an integral domain. I_3 is not maximal because $(0) \subsetneq (x) \subsetneq \Bbbk[x, y]$.

Proposition 2.12. *Let I be an ideal in the ring R.*

(1) *I is a prime ideal if and only if R/I is an integral domain. I is a maximal ideal if and only if R/I is a field.*

(2) *Every maximal ideal is prime. Every prime ideal is radical.*

Proof. (1) is non-examinable. (2) is an exercise. □

Under the corrspondence (2.2), we will find out what prime and maximal ideals correspond to. Now we switch to geometry.

Definition 2.13. An algebraic set $X \subseteq \mathbb{A}^n$ is *irreducible* if there does not exist a decomposition of X as a union of two stricly smaller algebraic sets. An irreducible (affine) algebraic set is also called an *affine variety*. An algebraic set $X \subseteq \mathbb{A}^n$ is *reducible* if it is not irreducible.

Example 2.14. We look at some algebraic sets in $\mathbb{A}^2$.

(1) The algebraic set $\mathbb{V}(xy) \subseteq \mathbb{A}^2$ is the union of two coordinate axes. In other words, $\mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$. Since each coordinate axis is an algebraic set stricly smaller than $\mathbb{V}(xy)$, we conclude that $\mathbb{V}(xy)$ is reducible.

(2) The algebraic set $\mathbb{V}(x, y) \subseteq \mathbb{A}^2$ consists of just one point, hence there is no way to decompose it as the union of two strictly smaller algebraic sets. It follows that $\mathbb{V}(x, y)$ is irreducible. Similarly, a point is always irreducible.

Next we show that prime ideals correspond to irreducible algebraic sets.

Proposition 2.15. *Let I be a radical ideal in $\Bbbk[x_1, \cdots, x_n]$ and $X = \mathbb{V}(I)$ the algebraic set in $\mathbb{A}^n$ defined by I. Then I is prime if and only if X is irreducible.*

Proof. In fact we prove the contrapositive: X is reducible $\Longleftrightarrow$ I is not prime.

We first prove "$\Longrightarrow$". Suppose $X = X_1 \cup X_2$ with algebraic sets $X_1, X_2 \subsetneq X$. Then $X_1 \subsetneq X$ implies that $\mathbb{I}(X_1) \supsetneq \mathbb{I}(X)$ by Proposition 2.9 (2). Hence there exists $f_1 \in \mathbb{I}(X_1) \backslash \mathbb{I}(X)$. Similarly $X_2 \subsetneq X$ implies that there exists $f_2 \in \mathbb{I}(X_2) \backslash \mathbb{I}(X)$. The product $f_1 f_2$ vanishes at all points of X, hence $f_1 f_2 \in \mathbb{I}(X)$. Therefore $I = \mathbb{I}(X)$ is not prime.

We then prove "$\Longleftarrow$". Since I is not prime, there exist $f_1, f_2 \notin I$ such that $f_1 f_2 \in I$. Consider the set $S_1 = I \cup \{f_1\}$. Then $X_1 = \mathbb{V}(S_1)$ is an algebraic set. Since $S_1 \supseteq I$, we have $X_1 \subseteq X$ by Proposition 1.7. Moreover, since $f_1 \notin I$, there is some point $p \in X$ such that $f_1(p) \neq 0$, therefore $p \notin X_1$. It follows that $X_1 \subsetneq X$. Similarly we can consider $S_2 = I \cup \{f_2\}$, then $X_2 = \mathbb{V}(S_2) \subsetneq X$.

It remains to show that $X_1 \cup X_2 = X$. Since X_1 and X_2 are subsets of X, we have $X_1 \cup X_2 \subseteq X$. Conversely, for any $p \in X$, $f(p) = 0$ for every $f \in I$. Moreover $f_1(p) f_2(p) = 0$, which implies $f_1(p) = 0$ or $f_2(p) = 0$. Therefore $p \in \mathbb{V}(S_1) = X_1$ or $p \in \mathbb{V}(S_2) = X_2$. This implies $X \subseteq X_1 \cup X_2$. $\square$

Finally we show that maximal ideals correspond to points.

Proposition 2.16. *Let I be a radical ideal in $\Bbbk[x_1, \cdots, x_n]$ and $X = \mathbb{V}(I)$ the algebraic set in $\mathbb{A}^n$ defined by I. Then I is maximal if and only if X is a point.*

Proof. (This proof is non-examinable and not covered in lectures.)

In fact we prove the contrapositive: X is not a point $\Longleftrightarrow$ I is not maximal.

We first prove "$\Longrightarrow$". If X is not a point, then either $X = \varnothing$ or X contains more than one point. If $X = \varnothing$, then by Proposition 2.9 (1), $I = \mathbb{I}(X) = \Bbbk[x_1, \cdots, x_n]$ is not a proper ideal hence not maximal. If X contains more than one point, then we can pick a subset Y of X containing only one point. Hence we have $\varnothing \subsetneq Y \subsetneq X$. By Proposition 2.9 (2), we have $\Bbbk[x_1, \cdots, x_n] = \mathbb{I}(\varnothing) \supsetneq \mathbb{I}(Y) \supsetneq \mathbb{I}(X)$. Hence $I = \mathbb{I}(X)$ is not maximal.

We then prove "$\Longleftarrow$". If I is not maximal, then either I is not a proper ideal, or there exists an ideal J such that $I \subsetneq J \subsetneq \Bbbk[x_1, \cdots, x_n]$. If I is not proper then $I = \Bbbk[x_1, \cdots, x_n]$, hence $X = \mathbb{V}(I) = \varnothing$ which is not a point. If $I \subsetneq J \subsetneq \Bbbk[x_1, \cdots, x_n]$ for some ideal J, then we claim that we actually have $I \subsetneq \sqrt{J} \subsetneq \Bbbk[x_1, \cdots, x_n]$. Indeed, by Lemma 2.2, we have $I \subsetneq J \subseteq \sqrt{J}$. Moreover, by Nullstellensatz 2.8 (1), we have $\mathbb{V}(J) \neq \varnothing$, hence $\sqrt{J} = \mathbb{I}(\mathbb{V}(J)) \subsetneq \Bbbk[x_1, \cdots, x_n]$. Armed with this claim we apply Proposition 2.9 (2) to get $\mathbb{V}(I) \supsetneq \mathbb{V}(\sqrt{J}) \supsetneq \varnothing$. It follows that $\mathbb{V}(\sqrt{J})$ contains at least one point, hence $X = \mathbb{V}(I)$ contains more than one point. $\square$

In summary, the $\mathbb{V} - \mathbb{I}$ correspondences induce bijections in each row of the diagram:

$$\begin{tikzcd}
\{\text{radical ideals in } \Bbbk[x_1, \cdots, x_n]\} \arrow[r, shift left, "\mathbb{V}"] & \{\text{algebraic sets in } \mathbb{A}^n\} \arrow[l, shift left, "\mathbb{I}"] \\
\{\text{prime ideals in } \Bbbk[x_1, \cdots, x_n]\} \arrow[u, hook] \arrow[r, shift left, "\mathbb{V}"] & \{\text{irreducible algebraic sets in } \mathbb{A}^n\} \arrow[l, shift left, "\mathbb{I}"] \arrow[u, hook] \\
\{\text{maximal ideals in } \Bbbk[x_1, \cdots, x_n]\} \arrow[u, hook] \arrow[r, shift left, "\mathbb{V}"] & \{\text{points in } \mathbb{A}^n\} \arrow[l, shift left, "\mathbb{I}"] \arrow[u, hook]
\end{tikzcd}$$