3. Coordinate Rings

We define polynomial functions and coordinate rings for algebraic sets. We will also study polynomial maps between algebraic sets. Finally we will see how coordinate rings help us understand polynomial maps.

3.1. Coordinate rings and polynomial maps.

We look at functions on affine algebraic sets. Roughly speaking, a function on an algebraic set X assigns to each point a value in $\Bbbk$. In algebraic geometry we are mostly interested in those functions defined by polynomials.

Definition 3.1. Let $X \subseteq \mathbb{A}^n$ be an algebraic set. A function $\varphi : X \to \Bbbk$ is a *polynomial function* if there exists $f \in \Bbbk[x_1, \cdots, x_n]$ such that $\varphi(p) = f(p)$ for every $p \in X$.

Remark 3.2. Two polynomials $f, g \in \Bbbk[x_1, \cdots, x_n]$ define the same function on X if and only if for every point $p \in X$, $f(p) = g(p)$, or equivalently, $f(p) - g(p) = 0$. This holds if and only if $f - g \in \mathbb{I}(X)$ by the definition of $\mathbb{I}$. In other words, f and g define the same polynomial function on X if and only if they are in the same coset of $\mathbb{I}(X)$ in $\Bbbk[x_1, \cdots, x_n]$. Therefore a polynomial function can be viewed as a coset of $\mathbb{I}(X)$, which is an element in the quotient ring $\Bbbk[x_1, \cdots, x_n]/\mathbb{I}(X)$. This leads to the following definition.

Definition 3.3. Let $X \subseteq \mathbb{A}^n$ be an algebraic set. The quotient ring

$$\Bbbk[X] := \Bbbk[x_1, \cdots, x_n]/\mathbb{I}(X)$$

is called the *coordinate ring* of X.

Example 3.4. For any algebraic set $X \subseteq \mathbb{A}^n$, the i-th coordinate defines a polynomial function $x_i : X \to \Bbbk$, which is called the i-th *coordinate function*. Since every polynomial function is a polynomial in the coordinate functions, we can view the coordinate functions as the generators of $\Bbbk[X]$. This is where the name "coordinate ring" comes from.

Example 3.5. For the algebraic set $X_1 = \mathbb{V}(x) \subseteq \mathbb{A}^2$, $\mathbb{I}(X_1) = (x)$ since (x) is a prime ideal hence is radical. Therefore the coordinate ring of X_1 is $\Bbbk[X_1] = \Bbbk[x, y]/(x)$. We show that it is isomorphic $\Bbbk[t]$. Consider the ring homomorphism

$$\varphi : \Bbbk[x, y] \to \Bbbk[t]; \quad x \mapsto 0, \quad y \mapsto t.$$

It is surjective because each $p(t) \in \Bbbk[t]$ is the image of $p(y) \in \Bbbk[x, y]$. For any $f(x, y) \in \Bbbk[x, y]$, by collecting all terms involving x, we can write it as $f(x, y) = xg(x, y) + h(y)$. Its image $\varphi(f(x, y)) = h(t)$. Hence $f \in \ker(\varphi)$ is equivalent to $h(y) = 0$, which is further equivalent to $f(x, y) \in (x)$. This shows $\ker(\varphi) = (x)$. By the fundamental isomorphism theorem, we get $\Bbbk[X_1] = \Bbbk[x, y]/(x) \cong \Bbbk[t]$.

Example 3.6. For the algebraic sets $X_2 = \mathbb{V}(y)$ and $X_3 = \mathbb{V}(y - x^2)$ in $\mathbb{A}^2$, we can similarly find out that $\Bbbk[X_2] = \Bbbk[x, y]/(y) \cong \Bbbk[t]$ and $\Bbbk[X_3] = \Bbbk[x, y]/(y - x^2) \cong \Bbbk[t]$. It is not a coincidence that X_1, X_2 and X_3 have isomorphic coordinate rings. We will explain this later.

Now we study maps between algebraic sets.

Definition 3.7. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets. A map $\varphi : X \to Y$ is a *polynomial map* if there exist polynomial functions $f_1, \cdots, f_m \in \Bbbk[X]$, such that $\varphi(p) = (f_1(p), \cdots, f_m(p)) \in Y$ for every point $p \in X$.

Notice that a polynomial function on X is the same as a polynomial map from X to $\mathbb{A}^1$.

Example 3.8. Let $X \subseteq \mathbb{A}^n$ be any algebraic set. The identity map $\mathrm{id}_X : X \to X$; $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n)$ is a polynomial map.

Example 3.9. Let $W \subseteq \mathbb{A}^2$ be any algebraic set and $X = \mathbb{A}^1$. Then $\varphi_0 : W \to X$; $(x, y) \mapsto xy$ is a polynomial map.

Example 3.10. Let $X = \mathbb{A}^1$. Let $Y_1 = \mathbb{V}(y - x^2)$, $Y_2 = \mathbb{V}(y^2 - x^3 - x^2)$ and $Y_3 = \mathbb{V}(y^2 - x^3)$ be algebraic sets in $\mathbb{A}^2$. Then $\varphi_1 : X \to Y_1$; $t \mapsto (t, t^2)$ is a polynomial map from X to Y_1, since the point (t, t^2) satisfies the defining equation of Y_1. Similarly, we can check that $\varphi_2 : X \to Y_2$; $t \mapsto (t^2 - 1, t^3 - t)$ and $\varphi_3 : X \to Y_3$; $t \mapsto (t^2, t^3)$ are both polynomial maps.

Remark 3.11. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$, $Z \subseteq \mathbb{A}^l$ be algebraic sets. Consider polynomial maps

$$\varphi = (f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)) : X \to Y,$$
$$\psi = (g_1(y_1, \cdots, y_m), \cdots, g_l(y_1, \cdots, y_m)) : Y \to Z.$$

We can compose them to get a new polynomial map

$$\psi \circ \varphi = (g_1(f_1, \cdots, f_m), \cdots, g_l(f_1, \cdots, f_m)) : X \to Z.$$

Example 3.12. To compose $\varphi_0 : W \to X$ in Example 3.9 and $\varphi_1 : X \to Y_1$ in Example 3.10, for any point $p = (x, y) \in W$, we have $(\varphi_1 \circ \varphi_0)(x, y) = \varphi_1(xy) = (xy, x^2y^2)$. Hence we get the polynomial map $\varphi_1 \circ \varphi_0 : W \to Y_1$; $(x, y) \mapsto (xy, x^2y^2)$.

We can now describe when two algebraic sets "look the same".

Definition 3.13. A polynomial map $\varphi : X \to Y$ between algebraic sets is an *isomorphism* if there exists a polynomial map $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. Two algebraic sets X and Y are *isomorphic* if there exists an isomorphism between them.

Example 3.14. We show that $\varphi_1 : X \to Y_1$ in Example 3.10 is an isomorphism. Let $\psi_1 : Y_1 \to X$; $(x, y) \mapsto x$. Then the composition $\psi_1 \circ \varphi_1 : X \to X$ is given by $t \mapsto (t, t^2) \mapsto t$. Hence $\psi_1 \circ \varphi_1 = \mathrm{id}_X$. The other composition $\varphi_1 \circ \psi_1 : Y_1 \to Y_1$ is given by $(x, y) \mapsto x \mapsto (x, x^2)$. For every point $(x, y) \in Y_1$, we have $y - x^2 = 0$, hence $(x, y) = (x, x^2)$. This shows $\varphi_1 \circ \psi_1 = \mathrm{id}_{Y_1}$. We conclude that $\varphi_1 : X \to Y_1$ (and $\psi_1 : Y_1 \to X$) is an isomorphism; in other words, X and Y_1 are isomorphic.

Remark 3.15. If a polynomial map $\varphi : X \to Y$ is an isomorphism, then it induces a bijection between the points in X and Y. However, it is important to note that the converse is not true. We will see a counter example next time.

We will see next time that the coordinate ring captures a lot of geometry of the algebraic set. In particular, whether two algebraic sets are isomorphic can be easily seen from their coordinate rings.