3.2. **Homomophisms of coordinate rings.** We introduce a terminology which will be very convenient in our discussion.

Definition 3.16. A *finitely generated* $\Bbbk$-*algebra* is a ring that is isomorphic to a quotient of a polynomial ring $\Bbbk[x_1, \cdots, x_n]/I$. A $\Bbbk$-*algebra homomorphism* $\varphi : \Bbbk[y_1, \cdots, y_m]/J \to \Bbbk[x_1, \cdots, x_n]/I$ is a ring homomorphism such that $\varphi(c + J) = c + I$ for every constant polynomial $c \in \Bbbk$.

Recall that a polynomial function can be viewed as a polynomial map to $\mathbb{A}^1$.

Definition 3.17. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets. Let $\varphi : X \to Y$ be a polynomial map and $g \in \Bbbk[Y]$ a polynomial function. The *pullback* of g along φ is the polynomial function $g \circ \varphi \in \Bbbk[X]$, denoted $\varphi^*(g)$.

The pullback map along φ sends a polynomial function on Y to a polynomial function on X. We show that it preserves the ring structure and constants.

Lemma 3.18. *For any polynomial map $\varphi : X \to Y$, the pullback map*

$$\varphi^* : \quad \Bbbk[Y] \to \Bbbk[X]; \quad g \mapsto g \circ \varphi$$

is a $\Bbbk$-algebra homomorphism.

Proof. We need to verify φ^* preserves addition, multiplication and constants. For any $g_1, g_2 \in \Bbbk[Y]$, we need to show $(g_1 + g_2) \circ \varphi = g_1 \circ \varphi + g_2 \circ \varphi$. Indeed, for any point $p \in X$, $((g_1+g_2)\circ\varphi)(p) = (g_1+g_2)(\varphi(p)) = g_1(\varphi(p))+g_2(\varphi(p)) = (g_1\circ\varphi)(p)+(g_2\circ\varphi)(p)$. Hence φ^* preserves addition. Replacing additions by multiplications shows that φ^* preserves multiplication. Now assume g is a constant function on Y, say, there exists some $c \in \Bbbk$ such that $g(q) = c$ for every $q \in Y$. Then $(g \circ \varphi)(p) = g(\varphi(p)) = c$ for every $p \in X$. Therefore $\varphi^*(g)$ is the constant function on X which takes the same value as g. □

Example 3.19. The polynomial map $\varphi : \mathbb{A}^1 \to Y = \mathbb{V}(y-x^2)(\subseteq \mathbb{A}^2)$; $t \mapsto (t, t^2)$ induces a $\Bbbk$-algebra homomorphism $\varphi^* : \Bbbk[Y] \to \Bbbk[\mathbb{A}^1]$, or more precisely, $\varphi : \Bbbk[x, y]/(y-x^2) \to \Bbbk[t]$. For any polynomial function $f(x, y)$ on Y, $\varphi^*(f) = f(t, t^2) \in \Bbbk[t]$. In particular, for the coordinate functions x and y on Y, we have $\varphi^*(x) = t$ and $\varphi^*(y) = t^2$. For more examples, the pullback of the polynomial function $x + y$ is $t + t^2$; the pullback of x^2y is t^4, and the pullback of $3x^3 + 5y + 1$ is $3t^3 + 5t^2 + 1$.

We have seen that every polynomial map $\varphi : X \to Y$ induces a $\Bbbk$-algebra homomorphism $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$. Next we show this is a one-to-one correspondence. This is the key property of the "pullback" construction.

Theorem 3.20. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets. For every $\Bbbk$-algebra homomorphism $\Phi : \Bbbk[Y] \to \Bbbk[X]$, there exists a unique polynomial map $\varphi : X \to Y$, such that $\Phi = \varphi^*$.*

Proof. (This proof is non-examinable and not covered in lectures.)

We show the existence. For every coordinate function $y_i \in \Bbbk[Y]$, by assumption $f_i = \Phi(y_i) \in \Bbbk[X]$ is a polynomial function on X. Since Φ is a $\Bbbk$-algebra homomorphis, for any polynomial function $g(y_1, \cdots, y_m) \in \Bbbk[Y]$, the image $\Phi(g) = g(f_1, \cdots, f_m) \in \Bbbk[X]$.

We consider the polynomial map $\varphi = (f_1, \cdots, f_m) : X \to \mathbb{A}^m$. To show it is a polynomial map to Y, it must be checked that $(f_1(p), \cdots, f_m(p)) \in Y$ for every $p \in X$; that is, it must be checked that $h(f_1(p), \cdots, f_m(p)) = 0$ for every polynomial $h \in \mathbb{I}(Y)$. Since h represents the zero function in $\Bbbk[Y]$, $\Phi(h)$ is also the zero function in $\Bbbk[X]$, hence $\Phi(h)(p) = 0$ for every $p \in X$. It follows that $h(f_1(p), \cdots, f_m(p)) = \Phi(h)(p) = 0$, as desired.

To show $\Phi = \varphi^*$, it remains to show that $\Phi(g) = \varphi^*(g)$ for every $g \in \Bbbk[Y]$. Indeed, for any $p \in X$, $\Phi(g)(p) = g(f_1(p), \cdots, f_m(p)) = g(\varphi(p)) = (g \circ \varphi)(p) = \varphi^*(g)(p)$. Hence $\Phi(g) = \varphi^*(g)$, as required. This finishes the existence.

For uniqueness, assume there is another polynomial map $\varphi' = (f'_1, \cdots, f'_m) : X \to Y$ such that $\Phi = (f')^*$. Then for each i, $f'_i = (\varphi')^*(y_i) = \Phi(y_i) = \varphi^*(y_i) = f_i$. Hence $\varphi' = \varphi$. This finishes the uniqueness. □

Remark 3.21. This theorem gives a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{polynomial maps} \\ \varphi : \ X \longrightarrow Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \Bbbk\text{-algebra homomorphisms} \\ \varphi^* : \ \Bbbk[Y] \longrightarrow \Bbbk[X] \end{array} \right\}.$$

An application of this result is the following criterion for isomorphisms.

Proposition 3.22. *A polynomial map* $\varphi : X \to Y$ *is an isomorphism if and only if* $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$ *is a ring isomorphism.*

Proof. (This proof is non-examinable and not covered in lectures.)

Assume $\varphi : X \to Y$ is an isomorphism, then there exists $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. By applying the pullback construction on both sides, we have $\varphi^* \circ \psi^* = (\psi \circ \varphi)^* = (\mathrm{id}_X)^* = \mathrm{id}_{\Bbbk[X]}$. Similarly we have $\psi^* \circ \varphi^* = \mathrm{id}_{\Bbbk[Y]}$. Therefore φ^* and ψ^* are mutually inverse ring homomorphisms. Hence $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$ is an isomorphism.

Assume $\varphi^* : \Bbbk[Y] \to \Bbbk[X]$ is a ring isomorphism, then there exists $\Psi : \Bbbk[X] \to \Bbbk[Y]$ such that $\varphi^* \circ \Psi = \mathrm{id}_{\Bbbk[X]}$ and $\Psi \circ \varphi^* = \mathrm{id}_{\Bbbk[Y]}$. By the existence in Theorem 3.20 we can write $\Psi = \psi^*$ for some polynomial map $\psi : Y \to X$. Therefore we have $(\psi \circ \varphi)^* = \varphi^* \circ \psi^* = \varphi^* \circ \Psi = \mathrm{id}_{\Bbbk[X]} = (\mathrm{id}_X)^*$. By the uniqueness in Theorem 3.20, we get $\psi \circ \varphi = \mathrm{id}_X$. Similarly we can get $\varphi \circ \psi = \mathrm{id}_Y$. Hence $\varphi : X \to Y$ is an isomorphism. □

This is a very powerful result as it allows us to show a certain polynomial map is an isomorphism without writing down another one going backwards. It can also be used to show a certain polynomial map is not an isomorphism, especially in some tricky situation where the map is actually bijective on points, as shown in the following example:

Example 3.23. We consider the polynomial map $\varphi : \mathbb{A}^1 \to X = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2; t \mapsto (t^2, t^3)$; see Example 3.10. One can show that it is bijective on points in $\mathbb{A}^1$ and X. However, one can also show that $\varphi^* : \Bbbk[X] \to \Bbbk[\mathbb{A}^1]$ is not an isomorphism of rings, hence φ is not an isomorphism of algebraic sets. We leave the details as an exercise.