4. Projective Algebraic Sets

Instead of affine spaces, it is more natural to study algebraic geometry in projective spaces. We first introduce projective spaces, then study projective algebraic sets. There is a similar projective Nullstellensatz and $\mathbb{V}-\mathbb{I}$ correspondence.

4.1. **Projective spaces.**

We will study algebraic geometry in projective spaces. We prefer projective spaces because results in projective spaces are usually nicer. One such example is that: two curves in $\mathbb{A}^2$ may or may not intersect each other. When they intersect, the number of intersection is not known until one solves the system of equations. However, in projective spaces $\mathbb{P}^2$, two curves always intersect, and the number of intersection points can be easily read off from their equations. In this lecture we will understand the projective space $\mathbb{P}^n$ from the following three different points of views:

- $\mathbb{P}^n$ is the set of 1-dimensional subspaces in $\mathbb{A}^{n+1}$ (definition);
- $\mathbb{P}^n$ is covered by $n+1$ subsets which are all $\mathbb{A}^n$ (aka from projective to affine);
- $\mathbb{P}^n$ is obtained by adding to $\mathbb{A}^n$ a "boundary at infinity", whose points correspond to "asymptotic directions" in $\mathbb{A}^n$ (aka from affine to projective).

Definition 4.1. For every integer $n \geqslant 0$, the *projective space* $\mathbb{P}^n_{\Bbbk}$ (or $\mathbb{P}^n$ if $\Bbbk$ is understood) of dimension n over a field $\Bbbk$ is the set of 1-dimensional vector subspaces in $\mathbb{A}^{n+1}_{\Bbbk}$.

Remark 4.2. Each point $a = (a_0, a_1, \cdots, a_n) \neq (0, 0, \cdots, 0)$ in $\mathbb{A}^{n+1}$ determines a 1-dimensional subspace. Two such points $a = (a_0, a_1, \cdots, a_n)$ and $b = (b_0, b_1, \cdots, b_n)$ define the same subspace if and only if there is some $\lambda \neq 0$ such that $b_i = \lambda a_i$ for each $0 \leqslant i \leqslant n$. We say two such points are equivalent, and write $a \sim b$. Then points in $\mathbb{P}^n$ can be identified with such equivalence classes. More precisely,

$$\mathbb{P}^n = \left(\mathbb{A}^{n+1} \backslash \{(0, \cdots, 0)\}\right) / \sim .$$

Definition 4.3. If a point $p \in \mathbb{P}^n$ is determined by $(a_0, a_1, \cdots, a_n) \in \mathbb{A}^{n+1} \backslash \{(0, \cdots, 0)\}$, we say that $a_0, a_1, \cdots, a_n$ are *homogeneous coordinates* of p, denoted $p = [a_0 : a_1 : \cdots : a_n]$.

Remark 4.4. The homoeneous coordinates of $p \in \mathbb{P}^n$ are only determined up to a non-zero scalar multiplication, so the i-th coordinate a_i is not a well-defined number. However, it is a well-defined notion to say whether a_i is zero or non-zero; and if $a_i \neq 0$, the ratios a_j/a_i are also well-defined (since they remain unchanged under equivalence).

We want to relate projective spaces to our familiar affine spaces, so that we can "visualise" them easily. There are two typical ways to do this.

Construction 4.5 (From projective to affine)**.** We will see how to find subsets in $\mathbb{P}^n$ which are affine spaces. For each $0 \leqslant i \leqslant n$, consider the subset

$$U_i = \{[a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n \mid a_i \neq 0\}.$$

Each point $p \in U_i$ can be written as

$$p = \left[\frac{a_0}{a_i} : \cdots : \frac{a_{i-1}}{a_i} : 1 : \frac{a_{i+1}}{a_i} : \cdots : \frac{a_n}{a_i}\right].$$

Since we insist that the i-th coordinate is 1, the other n coordinates are uniquely determined, which can be used to identify U_i with $\mathbb{A}^n$. Moreover, since every point in $\mathbb{P}^n$ has at least one non-zero homogeneous coordinate, it lies in at least one of the U_i's. This implies

$$\mathbb{P}^n = \cup_{i=0}^n U_i. \tag{4.1}$$

So $\mathbb{P}^n$ is covered by $n+1$ subsets, each of which looks just like $\mathbb{A}^n$.

Definition 4.6. Each subset $U_i = \{[a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n \mid a_i \neq 0\}$ of $\mathbb{P}^n$ is called a *standard affine chart* of $\mathbb{P}^n$. For every point $p = [a_0 : a_1 : \cdots : a_n] \in U_i$, the n-tuple $\left(\frac{a_0}{a_i}, \cdots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \cdots, \frac{a_n}{a_i}\right)$ are called the *non-homogeneous coordinates* of p with respect to U_i. The cover $\mathbb{P}^n = \cup_{i=0}^n U_i$ is called a *standard affine cover* of $\mathbb{P}^n$.

Example 4.7. $\mathbb{P}^1$ has two standard affine charts. The point $[2 : 3] \in \mathbb{P}^1$ has non-homogeneous coordinate $\frac{3}{2}$ with respect to U_0, and $\frac{2}{3}$ with respect to U_1. $\mathbb{P}^2$ has three standard affine charts. The point $[2 : 3 : 0] \in \mathbb{P}^2$ has non-homogeneous coordinates $(\frac{3}{2}, 0)$ with respect to U_0, and $(\frac{2}{3}, 0)$ with respect to U_1. This point is not in U_2 because the corresponding coordinate is 0.

Construction 4.8 (From affine to projective)**.** We will see how to get $\mathbb{P}^n$ by adding "points at infinity" to the affine space $\mathbb{A}^n$. We work with U_0 but each U_i works in the same way. The complement of U_0 in $\mathbb{P}^n$ is

$$H_0 = \mathbb{P}^n \backslash U_0 = \{[0 : a_1 : \cdots : a_n] \in \mathbb{P}^n\},$$

which can be identified with $\mathbb{P}^{n-1}$ as each point in H_0 is given by n homogeneous coordinates which are not simultaneously zero. Hence $\mathbb{P}^n$ can be decomposed into an affine space $U_0 \cong \mathbb{A}^n$ and a set of "points at infinity" $H_0 \cong \mathbb{P}^{n-1}$:

$$\mathbb{P}^n = U_0 \cup H_0 \cong \mathbb{A}^n \cup \mathbb{P}^{n-1}. \tag{4.2}$$

Now we explain why we can view points in H_0 as "asymptotic directions" of lines in $U_0 = \mathbb{A}^n$. This is best illustrated for $n = 2$, but works for any positive integer n.

Example 4.9. Consider two lines $\mathbb{V}(x_2 - x_1 + 1)$ and $\mathbb{V}(x_2 - x_1 - 1)$ in $\mathbb{A}^2 \cong U_0$. They are parallel since they have the same slope. We can regard x_1 and x_2 as the non-homogeneous coordinates with respect to U_0, and substitute x_i by $\frac{a_i}{a_0}$. Then the defining equations of the two lines become

$$\frac{a_2}{a_0} - \frac{a_1}{a_0} \pm 1 = 0.$$

We clear the denominators to get

$$a_2 - a_1 \pm a_0 = 0.$$

Notice that after clearing the denominator, we no longer require a_0 to be non-zero. Therefore we could possibly get extra solutions corresponding to points in H_0. To see which points in H_0 satisfy the equation, we set $a_0 = 0$. Then the equation becomes

$$a_2 - a_1 = 0.$$

Up to a non-zero scalar multiplication we get one extra solution $[a_0 : a_1 : a_2] = [0 : 1 : 1]$. So we can say both lines pass through (and intersect at) the point $[0 : 1 : 1]$ at infinity. Since parallel lines always acquire the same point at infinity, we get an idea that points in H_0 correspond to "asymptotic directions".

This example shows us how to understand points at infinity. We use the line $\mathbb{V}(x_2 - x_1 + 1)$ to preview some notions that will come up later. After clearing the denominators, we get a polynomial $a_2 - a_1 + a_0$ in which every monomial has the same degree. We say such a polynomial is *homogeneous*. Its solutions in $\mathbb{P}^2$ is called a *projective algebraic set*. Since it is obtained by adding the appropriate "points at infinity" to the affine algebraic set $\mathbb{V}(x_2 - x_1 + 1)$, we say this projective algebraic set is the *projective closure* of the affine algebraic set $\mathbb{V}(x_2 - x_1 + 1)$. In fact, every affine algebraic set in $\mathbb{A}^n$ (not necessarily a line) has a projective closure in $\mathbb{P}^n$ obtained by adding the appropiate "points at infinity", which can be computed using a similar calculation. We will see more examples later.