4.2. **Projective algebraic sets and projective Nullstellensatz.** We develop the theory of projective algebraic sets. Most of the results and proofs are similar to those in the affine case. We will be brief on the similar part, but careful on a few special features.

Definition 4.10. A non-zero polynomial $f \in \Bbbk[z_0, z_1, \cdots, z_n]$ is *homogeneous of degree d* if each term of f has the same total degree d.

As easy examples, $z_2 - z_1^2$ is not homogeneous while $z_0 z_2 - z_1^2$ is homogeneous of degree 2. The importance of this notion is the following. If f is homogeneous of degree d, then

$$f(\lambda a_0, \lambda a_1, \cdots, \lambda a_n) = \lambda^d f(a_0, a_1, \cdots, a_n). \tag{4.3}$$

In particular this means $f(\lambda a_0, \lambda a_1, \cdots, \lambda a_n) = 0$ if and only if $f(a_0, a_1, \cdots, a_n) = 0$ for any $\lambda \neq 0$. Therefore for any point $p = [a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n$, the condition $f(p) = 0$ is independent of the choice of its homogeneous coordinates. Hence the zero locus of f

$$\{[a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n \mid f(a_0, a_1, \cdots, a_n) = 0\}$$

is also well-defined.

Remark 4.11. Since the zero polynomial satisfies (4.3) for every non-negative integer d, as a convention, the zero polynomial is considered to be a homogeneous polynomial of any degree. By doing so, we can avoid many unnecessary exceptions. For instance, the sum of two homogeneous polynomial of degree d is again a homogeneous polynomial of degree d when we include the zero polynomial.

Definition 4.12. For any non-zero polynomial $f \in \Bbbk[z_0, z_1, \cdots, z_n]$ of degree m, we say $f = f_0 + f_1 + \cdots + f_m$ is the *homogeneous decomposition* of f, if for each i, $0 \leqslant i \leqslant m$, f_i is homogeneous of degree i. Each f_i is called a *homogeneous component* of f.

Definition 4.13. An ideal $I \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$ is *homogeneous* if for every non-zero polynomial $f \in I$, each of its homogeneous components $f_i \in I$.

In practice, this condition for an ideal being homogeneous is not very easy to check. The following criterion is usually more convenient.

Proposition 4.14. *An ideal $I \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$ is homogeneous if and only if it can be generated by a finite set of homogeneous polynomials.*

Proof. We leave the proof as an exercise. □

Example 4.15. The ideals (x) and (x, y^2) in $\Bbbk[x, y]$ are both homogeneous, while the ideal $(x+y^2)$ in $\Bbbk[x, y]$ is not homogeneous, because the degree 1 part of $x+y^2$ is x, which is not in this ideal.

Notice that an ideal could have many different sets of generators. The statement only requires one set of generators consists of only homogeneous polynomials. It is still possible that some other generating set is not given by homogeneous polynomials. Next we can define the correspondences $\mathbb{V}$ and $\mathbb{I}$.

Definition 4.16. For any homogeneous ideal $I \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$, the set

$$\mathbb{V}(I) = \{p \in \mathbb{P}^n \mid f(p) = 0 \text{ for every homogeneous polynomial } f \in I\}$$

is called the *projective algebraic set* defined by I.

Similar to the affine case, the following result is usually convenient in practice.

Lemma 4.17. *Suppose a homogeneous ideal $I \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$ is generated by a finite set of homogeneous polynomials $S = \{f_1, \cdots, f_m\}$. Let*

$$\mathbb{V}(S) = \{p \in \mathbb{P}^n \mid f_1(p) = \cdots f_m(p) = 0\}.$$

Then $\mathbb{V}(S) = \mathbb{V}(I)$.

Proof. Similar to the proof of Lemma 1.10. We leave it as an exercise. □

Corollary 4.18. *Every projective algebraic set $X \subseteq \mathbb{P}^n$ can be written as $\mathbb{V}(S)$ for a finite set S of homogeneneous polynomials in $\Bbbk[z_0, \cdots, z_n]$.*

Proof. It follows immediately from Propositions 4.14 and 4.17. □

Example 4.19. In $\mathbb{P}^1$, the projective algebraic set $\mathbb{V}(3z_0 - 2z_1)$ is the single-point set $\{[2:3]\}$. In $\mathbb{P}^2$, the projective algebraic set $\mathbb{V}(z_2 - z_1 + z_0)$ is one of the affine lines in Example 4.9 together with the corresponding point at infinity.

Definition 4.20. A projective algebraic set $X \subseteq \mathbb{P}^n$ is called a *hypersurface* if $X = \mathbb{V}(f)$ for some non-constant homogeneous polynomial $f \in \Bbbk[z_0, z_1, \cdots, z_n]$.

Definition 4.21. For any subset $X \subseteq \mathbb{P}^n$, the set

$$\mathbb{I}(X) = \left\{ f \in \Bbbk[z_0, z_1, \cdots, z_n] \;\middle|\; \begin{array}{c} f(p) = 0 \text{ for every choice of homogeneous} \\ \text{coordinates of every point } p \in X \end{array} \right\}$$

is called the *ideal of X*.

Lemma 4.22. *For any subset $X \subseteq \mathbb{P}^n$, $\mathbb{I}(X)$ is a homogeneous radical ideal.*

Proof. The proof of Lemma 2.6 (2) works literally here to show $\mathbb{I}(X)$ is a radical ideal. To show it is homogeneous, let $f \in \mathbb{I}(X)$ and write $f = f_0 + f_1 + \cdots + f_m$ for the homogeneous decomposition of f where m is the degree of f. For each $p = [a_0 : a_1 : \cdots : a_n] \in X$ and $\lambda \in \mathbb{k}\backslash\{0\}$, we can also write $p = [\lambda a_0 : \lambda a_1 : \cdots : \lambda a_n]$, hence we have

$$\begin{aligned} 0 = f(p) &= f(\lambda a_0, \lambda a_1, \cdots, \lambda a_n) \\ &= \sum_{i=0}^{m} f_i(\lambda a_0, \lambda a_1, \cdots, \lambda a_n) \\ &= \sum_{i=0}^{m} \lambda^i f_i(a_0, a_1, \cdots, a_n) = \sum_{i=0}^{m} \lambda^i f_i(p). \end{aligned}$$

This means that every $\lambda \in \mathbb{k}\backslash\{0\}$ is a root of the polynomial $\sum_{i=0}^{m} f_i(p)x^i \in \mathbb{k}[x]$. This must be a zero polynomial, because the number of roots of any non-zero polynomial is at most equal to its degree m. It follows that $f_i(p) = 0$ for every $0 \leqslant i \leqslant m$, so $f_i \in \mathbb{I}(X)$. □

Remark 4.23. We have used the same notation $\mathbb{V}$ and $\mathbb{I}$ in both affine and projective cases. In practice it is usually clear which is meant; but if there is any danger of confusion, we will write $\mathbb{V}_p$ and $\mathbb{I}_p$ for the projective operations, $\mathbb{V}_a$ and $\mathbb{I}_a$ for the affine ones.

Now we state the projective Nullstellensatz. It is similar to the affine version, but there is one point where care is needed. Clearly the trivial ideal $(1) = \mathbb{k}[z_0, z_1, \cdots, z_n]$ defines the empty set in $\mathbb{A}^{n+1}$, hence the empty set in $\mathbb{P}^n$, as it should be. However, the ideal $(z_0, z_1, \cdots, z_n)$ defines a single-point set $\{(0, \cdots, 0)\}$ in $\mathbb{A}^{n+1}$, which also corresponds to the empty set in $\mathbb{P}^n$. This ideal $(z_0, z_1, \cdots, z_n)$ is an awkward exception to several statements in the theory, and is traditionally known as the "irrelevant ideal". Keeping that in mind, we state the projective version of Nullstellensatz.

Theorem 4.24 (Projective Nullstellensatz)**.** *Let $\mathbb{k}$ be an algebraically closed field. For any homogeneous ideal $I \subseteq \mathbb{k}[z_0, z_1, \cdots, z_n]$,*

(1) *$\mathbb{V}(I) = \varnothing$ if and only if $\sqrt{I} \supseteq (z_0, z_1, \cdots, z_n)$.*
(2) *If $\mathbb{V}(I) \neq \varnothing$, then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.*

Proof. This is an easy consequence of the affine Nullstellensatz. Non-examinable. Interested reader can find the proof in [Section 5.3, Reid, Undergraduate Algebraic Geometry] or [Section 4.2, Fulton, Algebraic Curves]. □