5. Rational Maps

We have seen projective algebraic sets. Now we study $\mathbb{V}-\mathbb{I}$ correspondence for projective algebraic sets and maps between them.

5.1. $\mathbb{V}-\mathbb{I}$ correspondence and rational maps.

We have introduced the projective Nullstellensatz. The following notion is parallel to the same one in the affine case.

Definition 5.1. A projective algebraic set $X \subseteq \mathbb{P}^n$ is *irreducible* if there does not exist a decomposition of X as a union of two stricly smaller projective algebraic sets. An irreducible projective algebraic set is also called an *projective variety.* A projective algebraic set $X \subseteq \mathbb{P}^n$ is *reducible* if it is not irreducible.

Not very surprisingly, we also have the projective version of $\mathbb{V}-\mathbb{I}$ correspondences. Each row in the following diagram is a bijection:

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{homogeneous radical ideals}\\ I \subseteq \mathbb{k}[z_0, z_1, \cdots, z_n]\\ \text{with } I \not\supseteq (z_0, z_1, \cdots, z_n)\end{array}\right\} & \underset{\mathbb{I}}{\overset{\mathbb{V}}{\rightleftarrows}} & \left\{\begin{array}{c}\text{non-empty projective}\\ \text{algebraic sets } X \subseteq \mathbb{P}^n\end{array}\right\} \\
\uparrow & & \uparrow \\
\left\{\begin{array}{c}\text{homogeneous prime ideals}\\ I \subseteq \mathbb{k}[z_0, z_1, \cdots, z_n]\\ \text{with } I \not\supseteq (z_0, z_1, \cdots, z_n)\end{array}\right\} & \underset{\mathbb{I}}{\overset{\mathbb{V}}{\rightleftarrows}} & \left\{\begin{array}{c}\text{non-empty}\\ \text{irreducible projective}\\ \text{algebraic sets } X \subseteq \mathbb{P}^n\end{array}\right\}
\end{array}$$

We summarise the content in the diagram in words for later reference.

Proposition 5.2. *Let X be a non-empty projective algebraic set in $\mathbb{P}^n$ and I a homogeneous radical ideal in $\mathbb{k}[z_0, \cdots, z_n]$ such that $(z_0, \cdots, z_n) \not\subseteq I$. Then $X = \mathbb{V}(I)$ if and only if $I = \mathbb{I}(X)$. In such a case, X is irreducible if and only if I is prime.*

Proof. Non-examinable. Interested reader can find the proof in [Section 5.3, Reid, Undergraduate Algebraic Geometry]. □

Remark 5.3. Comparing with the affine $\mathbb{V}-\mathbb{I}$ correspondence, the bijection between maximal ideals and points is no longer valid in the projective setting. In fact, the only homogeneous maximal ideal in $\mathbb{k}[z_0, z_1, \cdots, z_n]$ is the irrelevant ideal $(z_0, z_1, \cdots, z_n)$, which gives the empty set in $\mathbb{P}^n$ as we discussed above.

In practice it is usually not easy to determine whether a projective algebraic set is irreducible. It is clear that $\mathbb{P}^n$ is irreducible since $\mathbb{I}(\mathbb{P}^n) = (0)$ is a prime ideal. In case of hypersurfaces, the following result usually helps.

Lemma 5.4. *Let $I = (f) \subseteq \Bbbk[z_0, z_1, \cdots, z_n]$. Then I is a prime ideal if and only if f is an irreducible polynomial; I is a radical ideal if and only if f has no repeated irreducible factors.*

Proof. It was proved in Exercise 2.2. □

Now we turn to maps between projective algebraic sets.

Definition 5.5. For projective algebraic sets $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, a *rational map* $\varphi : X \dashrightarrow Y$ is an equivalence class of expressions $[f_0 : \cdots : f_m]$ satisfying

(1) $f_0, \cdots, f_m \in \Bbbk[z_0, \cdots, z_n]$ are homogeneous of the same degree;

(2) $[f_0(p) : \cdots : f_m(p)] \neq [0 : \cdots : 0]$ for some point $p \in X$;

(3) For each point $p \in X$, if $[f_0(p) : \cdots : f_m(p)]$ is defined, then it is a point in Y.

Two such expressions $[f_0 : \cdots : f_m]$ and $[g_0 : \cdots : g_m]$ are equivalent if $[f_0(p) : \cdots : f_m(p)] = [g_0(p) : \cdots : g_m(p)]$ for every $p \in X$ at which both are defined.

Definition 5.6. Let $\varphi : X \dashrightarrow Y$ be a rational map between projective algebraic sets. We say φ is *regular* at $p \in X$ if $[f_0(p) : \cdots : f_m(p)]$ is well-defined for some expression $[f_0 : \cdots : f_m]$ representing φ.

Definition 5.7. For projective algebraic sets X and Y, a *morphism* $\varphi : X \longrightarrow Y$ is a rational map which is regular at every point in X.

Remark 5.8. The condition (1) in Definition 5.5 guarantees that the image is independent of the choice of the homogeneous coordinates of p. More precisely, suppose f_i's are homogeneous of degree d, and $p = [a_0 : \cdots : a_n]$. For any $\lambda \neq 0$, we can also write $p = [\lambda a_0 : \cdots : \lambda a_n]$. Then we have by (4.3) that

$$\begin{aligned} &[f_0(\lambda a_0, \cdots, \lambda a_n) : \cdots : f_m(\lambda a_0, \cdots, \lambda a_n)] \\ =& [\lambda^d f_0(a_0, \cdots, a_n) : \cdots : \lambda^d f_m(a_0, \cdots, a_n)] \\ =& [f_0(a_0, \cdots, a_n) : \cdots : f_m(a_0, \cdots, a_n)]. \end{aligned}$$

The condition (2) in Definition 5.5 guarantees that the expression $[f_0 : \cdots : f_m]$ is defined on a non-empty subset of X.

Remark 5.9. We can view a rational function $\varphi : X \dashrightarrow Y$ as a piecewise and partially defined function. Each expression $[f_0 : \cdots : f_m]$ representing φ is defined on a subset of X. Two such expressions that agree on the locus where both are defined can be glued together to represent the same function φ. However, there could still be some points in X where none of the expressions is defined.

Example 5.10. We check the following is a morphism

$$\varphi: \quad \mathbb{P}^1 \longrightarrow \mathbb{P}^2; \quad [u:v] \longmapsto [u^2 : uv : v^2].$$

All components of φ are homogeneous polynomials of degree 2. For each point $p = [u : v] \in \mathbb{P}^1$, either $u \neq 0$ or $v \neq 0$, hence either $u^2 \neq 0$ or $v^2 \neq 0$. Therefore φ is regular on the entire $\mathbb{P}^1$. Since the target is $\mathbb{P}^2$, $\varphi(p) \in \mathbb{P}^2$ is automatic for every $p \in \mathbb{P}^1$.

Example 5.11. Consider the projective algebraic set $C = \mathbb{V}(z_0 z_2 - z_1^2) \subseteq \mathbb{P}^2$. We check the following is a morphism

$$\varphi: \quad \mathbb{P}^1 \longrightarrow C; \quad [u:v] \longmapsto [u^2 : uv : v^2].$$

We need to check everything that we checked in Example 5.10. In addition we need to check $\varphi(p) \in C$ for every $p \in \mathbb{P}^1$. To see that we need to show $[u^2 : uv : v^2]$ satisfies the defining equation of C, which is clear since $(u^2)(v^2) - (uv)^2 = 0$.

Example 5.12. For the same C as in Example 5.11, we check the following is a morphism

$$\psi: \quad C \longrightarrow \mathbb{P}^1; \quad [z_0 : z_1 : z_2] \longmapsto \begin{cases} [z_0 : z_1] & \text{if } z_0 \neq 0; \\ [z_1 : z_2] & \text{if } z_2 \neq 0. \end{cases}$$

As we can see ψ is defined by two expressions, whose components are all homogeneous polynomials of degree 1. They are both defined on a non-empty subset of C; e.g. both are defined at $[1 : 1 : 1] \in C$. It is clear that the image is always in $\mathbb{P}^1$. For any point $[z_0 : z_1 : z_2] \in C$ with $z_0 \neq 0$ and $z_2 \neq 0$, we have $z_1^2 = z_0 z_2$ hence $z_1 \neq 0$. Set $\lambda = \frac{z_1}{z_0} = \frac{z_2}{z_1} \neq 0$, then $[z_0 : z_1] = [\lambda z_0 : \lambda z_1] = [z_1 : z_2]$. Therefore the two expressions agree on the locus where they are both defined. To show ψ is regular everywhere on C, we observe that for any point $p = [z_0 : z_1 : z_2] \in C$, z_0 and z_2 cannot be both zero, since otherwise $z_1^2 = z_0 z_2 = 0$ and p is not a valid point. This concludes that ψ is a morphism.

Example 5.13 (Cremona transformation)**.** We check the following is a rational map

$$\varphi: \quad \mathbb{P}^2 \dashrightarrow \mathbb{P}^2; \quad [x:y:z] \longmapsto [yz : zx : xy].$$

All components of φ are homogeneous of degree 2. For every point $p \in \mathbb{P}^2$ with at least two non-zero coordinates, $\varphi(p)$ is a well-defined point in $\mathbb{P}^2$. Hence φ is a rational map.