5. Rational Maps

We have seen projective algebraic sets. Now we study $\mathbb{V} - \mathbb{I}$ correspondence for projective algebraic sets and maps between them.

5.1. $\mathbb{V} - \mathbb{I}$ correspondence and rational maps. We have introduced the projective Nullstellensatz. The following notion is parallel to the same one in the affine case.

**Definition 5.1.** A projective algebraic set $X \subseteq \mathbb{P}^n$ is **irreducible** if there does not exist a decomposition of $X$ as a union of two strictly smaller projective algebraic sets. An irreducible projective algebraic set is also called an **projective variety**. A projective algebraic set $X \subseteq \mathbb{P}^n$ is **reducible** if it is not irreducible.

Not very surprisingly, we also have the projective version of $\mathbb{V} - \mathbb{I}$ correspondences. Each row in the following diagram is a bijection:

\[
\begin{align*}
\{ & \text{homogeneous radical ideals} \\
& I \subseteq k[z_0, z_1, \cdots, z_n] \\
& \text{with } I \not\supseteq (z_0, z_1, \cdots, z_n) \} & \xymatrix{ \mathbb{V} \ar[r] & \mathbb{I} } & \{ & \text{non-empty projective algebraic sets } X \subseteq \mathbb{P}^n \} \\
\downarrow \downarrow \downarrow & & & \downarrow \downarrow \downarrow & \{ & \text{homogeneous prime ideals} \\
& I \subseteq k[z_0, z_1, \cdots, z_n] \\
& \text{with } I \not\supseteq (z_0, z_1, \cdots, z_n) \} & \xymatrix{ \mathbb{V} \ar[r] & \mathbb{I} } & \{ & \text{non-empty irreducible projective algebraic sets } X \subseteq \mathbb{P}^n \}
\end{align*}
\]

We summarise the content in the diagram in words for later reference.

**Proposition 5.2.** Let $X$ be a non-empty projective algebraic set in $\mathbb{P}^n$ and $I$ a homogeneous radical ideal in $k[z_0, \cdots, z_n]$ such that $(z_0, z_1, \cdots, z_n) \not\subseteq I$. Then $X = \mathbb{V}(I)$ if and only if $I = \mathbb{I}(X)$. In such a case, $X$ is irreducible if and only if $I$ is prime.

**Proof.** Non-examinable. Interested reader can find the proof in [Section 5.3, Reid, Undergraduate Algebraic Geometry]. \qed

**Remark 5.3.** Comparing with the affine $\mathbb{V} - \mathbb{I}$ correspondence, the bijection between maximal ideals and points is no longer valid in the projective setting. In fact, the only homogeneous maximal ideal in $k[z_0, z_1, \cdots, z_n]$ is the irrelevant ideal $(z_0, z_1, \cdots, z_n)$, which gives the empty set in $\mathbb{P}^n$ as we discussed above.

In practice it is usually not easy to determine whether a projective algebraic set is irreducible. It is clear that $\mathbb{P}^n$ is irreducible since $\mathbb{I}(\mathbb{P}^n) = (0)$ is a prime ideal. In case of hypersurfaces, the following result usually helps.
Lemma 5.4. Let $I = (f) \subseteq \mathbb{k}[z_0, z_1, \cdots, z_n]$. Then $I$ is a prime ideal if and only if $f$ is an irreducible polynomial; $I$ is a radical ideal if and only if $f$ has no repeated irreducible factors.

Proof. It was proved in Exercise 2.2. □

Now we turn to maps between projective algebraic sets.

Definition 5.5. For projective algebraic sets $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, a rational map $\varphi : X \dasharrow Y$ is an equivalence class of expressions $[f_0 : \cdots : f_m]$ satisfying

1. $f_0, \cdots, f_m \in \mathbb{k}[z_0, \cdots, z_n]$ are homogeneous of the same degree;
2. $[f_0(p) : \cdots : f_m(p)] \neq [0 : \cdots : 0]$ for some point $p \in X$;
3. For each point $p \in X$, if $[f_0(p) : \cdots : f_m(p)]$ is defined, then it is a point in $Y$.

Two such expressions $[f_0 : \cdots : f_m]$ and $[g_0 : \cdots : g_m]$ are equivalent if $[f_0(p) : \cdots : f_m(p)] = [g_0(p) : \cdots : g_m(p)]$ for every $p \in X$ at which both are defined.

Definition 5.6. Let $\varphi : X \dasharrow Y$ be a rational map between projective algebraic sets. We say $\varphi$ is regular at $p \in X$ if $[f_0(p) : \cdots : f_m(p)]$ is well-defined for some expression $[f_0 : \cdots : f_m]$ representing $\varphi$.

Definition 5.7. For projective algebraic sets $X$ and $Y$, a morphism $\varphi : X \rightarrow Y$ is a rational map which is regular at every point in $X$.

Remark 5.8. The condition (1) in Definition 5.5 guarantees that the image is independent of the choice of the homogeneous coordinates of $p$. More precisely, suppose $f_i$’s are homogeneous of degree $d$, and $p = [a_0 : \cdots : a_n]$. For any $\lambda \neq 0$, we can also write $p = [\lambda a_0 : \cdots : \lambda a_n]$. Then we have by (4.3) that

$$[f_0(\lambda a_0, \cdots, \lambda a_n) : \cdots : f_m(\lambda a_0, \cdots, \lambda a_n)] = [\lambda^d f_0(a_0, \cdots, a_n) : \cdots : \lambda^d f_m(a_0, \cdots, a_n)] = [f_0(a_0, \cdots, a_n) : \cdots : f_m(a_0, \cdots, a_n)].$$

The condition (2) in Definition 5.5 guarantees that the expression $[f_0 : \cdots : f_m]$ is defined on a non-empty subset of $X$.

Remark 5.9. We can view a rational function $\varphi : X \dasharrow Y$ as a piecewise and partially defined function. Each expression $[f_0 : \cdots : f_m]$ representing $\varphi$ is defined on a subset of $X$. Two such expressions that agree on the locus where both are defined can be glued together to represent the same function $\varphi$. However, there could still be some points in $X$ where none of the expressions is defined.
Example 5.10. We check the following is a morphism
\[ \varphi : \mathbb{P}^1 \to \mathbb{P}^2; \quad [u : v] \mapsto [u^2 : uv : v^2]. \]
All components of \( \varphi \) are homogeneous polynomials of degree 2. For each point \( p = [u : v] \in \mathbb{P}^1 \), either \( u \neq 0 \) or \( v \neq 0 \), hence either \( u^2 \neq 0 \) or \( v^2 \neq 0 \). Therefore \( \varphi \) is regular on the entire \( \mathbb{P}^1 \). Since the target is \( \mathbb{P}^2 \), \( \varphi(p) \in \mathbb{P}^2 \) is automatic for every \( p \in \mathbb{P}^1 \).

Example 5.11. Consider the projective algebraic set \( C = \mathcal{V}(z_0z_2 - z_1^2) \subseteq \mathbb{P}^2 \). We check the following is a morphism
\[ \varphi : \mathbb{P}^1 \to C; \quad [u : v] \mapsto [u^2 : uv : v^2]. \]
We need to check everything that we checked in Example 5.10. In addition we need to check \( \varphi(p) \in C \) for every \( p \in \mathbb{P}^1 \). To see that we need to show \( [u^2 : uv : v^2] \) satisfies the defining equation of \( C \), which is clear since \( (u^2)(v^2) - (uv)^2 = 0 \).

Example 5.12. For the same \( C \) as in Example 5.11, we check the following is a morphism
\[ \psi : C \to \mathbb{P}^1; \quad [z_0 : z_1 : z_2] \mapsto \begin{cases} [z_0 : z_1] & \text{if } z_0 \neq 0; \\ [z_1 : z_2] & \text{if } z_2 \neq 0. \end{cases} \]
As we can see \( \psi \) is defined by two expressions, whose components are all homogeneous polynomials of degree 1. They are both defined on a non-empty subset of \( C \); e.g. both are defined at \([1 : 1 : 1] \in C \). It is clear that the image is always in \( \mathbb{P}^1 \). For any point \([z_0 : z_1 : z_2] \in C \) with \( z_0 \neq 0 \) and \( z_2 \neq 0 \), we have \( z_1^2 = z_0z_2 \) hence \( z_1 \neq 0 \). Set \( \lambda = \frac{z_1}{z_0} = \frac{z_2}{z_1} \neq 0 \), then \([z_0 : z_1] = [\lambda z_0 : \lambda z_1] = [z_1 : z_2] \). Therefore the two expressions agree on the locus where they are both defined. To show \( \psi \) is regular everywhere on \( C \), we observe that for any point \( p = [z_0 : z_1 : z_2] \in C \), \( z_0 \) and \( z_2 \) cannot be both zero, since otherwise \( z_1^2 = z_0z_2 = 0 \) and \( p \) is not a valid point. This concludes that \( \psi \) is a morphism.

Example 5.13 (Cremona transformation). We check the following is a rational map
\[ \varphi : \mathbb{P}^2 \to \mathbb{P}^2; \quad [x : y : z] \mapsto [yz : zx : xy]. \]
All components of \( \varphi \) are homogeneous of degree 2. For every point \( p \in \mathbb{P}^2 \) with at least two non-zero coordinates, \( \varphi(p) \) is a well-defined point in \( \mathbb{P}^2 \). Hence \( \varphi \) is a rational map.