5.2. **Dominant rational maps and birational maps.** We have seen rational maps between projective algebraic sets. We now consider the composition of two rational maps. Suppose $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are rational maps. It is not always true that they can be composed to get $g \circ f : X \dashrightarrow Z$, because the image of f could be disjoint from the locus where g is defined. We will deal with this problem.

Definition 5.14. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be projective varieties. A rational map $\varphi : X \dashrightarrow Y$ is *dominant* if there does not exist a projective algebraic set $W \subsetneq Y$, such that $\varphi(p) \in W$ for every $p \in X$ where φ is defined.

Example 5.15. We claim the morphism $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ in Example 5.10 is not dominant. To see this, we consider $W = \mathbb{V}(z_0 z_2 - z_1^2) \subset \mathbb{P}^2$. We see that $W \subsetneq \mathbb{P}^2$ because $[1 : 1 : 0] \in \mathbb{P}^2 \backslash W$. But for every $p \in \mathbb{P}^1$, $\varphi(p) \in W$ because $(u^2)(v^2) - (uv)^2 = 0$.

The definition is handy for showing a rational map is not dominant. The following criterion is usually more convenient for showing a rational map is dominant.

Lemma 5.16. *Let $\varphi : X \dashrightarrow Y$ be a rational map between projective varieties. Suppose there exists a projective algebraic set $Z \subsetneq Y$, such that every $q \in Y \backslash Z$ can be written as $q = \varphi(p)$ for some $p \in X$. Then φ is dominant.*

Proof. Suppose on the contrary that there exists some projective algebraic set $W \subsetneq Y$ such that $\varphi(p) \in W$ for every $p \in X$ at which φ is defined. It is clear $Y \supseteq W \cup Z$. For every $q \in Y$, if $q = \varphi(p)$ for some $p \in X$, then $q \in W$; otherwise $q \in Z$. It follows that $Y \subseteq W \cup Z$. Therefore $Y = W \cup Z$ where both W and Z are projective algebraic sets strictly smaller than Y. This contradicts the irreducibility of Y. ☐

Remark 5.17. In explicit examples there are usually many possible choices for W in Definition 5.14 and Z in Lemma 5.16. You can choose the one that you find easy to use.

Example 5.18. We consider the morphism $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in Example 5.13. We know $\mathbb{P}^2$ is a projective variety. We claim φ is dominant. If not, then we can find a projective algebraic set $W \subsetneq \mathbb{P}^2$, such that $\varphi(p) \in W$ for every $p \in \mathbb{P}^2$ at which φ is defined.

We observe that the projective algebraic set $Z = \mathbb{V}(xyz)$ consists of all points in $\mathbb{P}^2$ with at least one zero coordinate, so $Z \subsetneq \mathbb{P}^2$. For every point $[a : b : c] \in \mathbb{P}^2 \backslash Z$, all coordinates are non-zero. It is in the image of φ since

$$\varphi([bc : ca : ab]) = [a^2bc : ab^2c : abc^2] = [a : b : c].$$

It follows from Lemma 5.16 that φ is dominant.

Now we answer the question asked at the beginning and give a sufficient condition for the existence of compositions.

Lemma 5.19. *Let $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ be rational maps between projective varieties. If φ is dominant, then $\psi \circ \varphi : X \dashrightarrow Z$ is a rational map.*

Proof. Non-examinable. Interested reader can find more details in [Section 4.10, Reid, Undergraduate Algebraic Geometry]. □

The following is another special class of rational maps.

Definition 5.20. Let $\varphi : X \dashrightarrow Y$ be a rational map between projective varieties. It is said to be a *birational map* if there exists another rational map $\psi : Y \dashrightarrow X$, such that $\psi \circ \varphi$ is a well-defined rational map equivalent to the identity map on X, and $\varphi \circ \psi$ is a well-defined rational map equivalent to the identity map on Y. We say a birational map φ is an isomorphism if both φ and ψ can be chosen to be morphisms.

Remark 5.21. More precisely, the condition that $\psi \circ \varphi$ is equivalent to id_X means that $(\psi \circ \varphi)(p) = p$ for every point $p \in X$ at which $\psi \circ \varphi$ is defined. A similar condition holds for the other composition $\varphi \circ \psi$.

Example 5.22. We claim that the rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ discussed in Examples 5.13 and 5.18 is a birational map. Let ψ be the same rational map as φ, then the composition $\psi \circ \varphi$ is given by the expression

$$(\psi \circ \varphi)([x : y : z]) = \psi([yz : zx : xy]) = [x^2yz : xy^2z : xyz^2].$$

For any point $[x : y : z]$ with all coordinates nonzero, we have $(\psi \circ \varphi)([x : y : z]) = [x^2yz : xy^2z : xyz^2] = [x : y : z]$. The same is true for $\varphi \circ \psi$. Therefore the claim holds.

Example 5.23. We claim that the morphism $\varphi : \mathbb{P}^1 \longrightarrow C$ in Example 5.11 is an isomorphism, with an inverse ψ given by the morphism in Example 5.12. For any $[u : v] \in \mathbb{P}^1$, either $u \neq 0$ or $v \neq 0$. If $u \neq 0$, then $u^2 \neq 0$, hence

$$(\psi \circ \varphi)([u : v]) = \psi([u^2 : uv : v^2]) = [u^2 : uv] = [u : v].$$

If $v \neq 0$, then $v^2 \neq 0$. We can similarly have

$$(\psi \circ \varphi)([u : v]) = \psi([u^2 : uv : v^2]) = [uv : v^2] = [u : v].$$

For the other composition, take any point $[z_0 : z_1 : z_2] \in C$. We showed in Example 5.12 that either $z_0 \neq 0$ or $z_2 \neq 0$. If $z_0 \neq 0$, then

$$(\varphi \circ \psi)([z_0 : z_1 : z_2]) = \varphi([z_0 : z_1]) = [z_0^2 : z_0z_1 : z_1^2] = [z_0^2 : z_0z_1 : z_0z_2] = [z_0 : z_1 : z_2].$$

If $z_0 \neq 0$, we can similarly have

$$(\varphi \circ \psi)([z_0 : z_1 : z_2]) = \varphi([z_1 : z_2]) = [z_1^2 : z_1z_2 : z_2^2] = [z_0z_2 : z_1z_2 : z_2^2] = [z_0 : z_1 : z_2].$$

Therefore both compositions are equivalent to identity maps hence φ is a rational map. Since φ and ψ are both morphisms, φ is in fact an isomorphism.

Definition 5.24. Two projective varieties X and Y are said to be *birational* if there exists a birational map $\varphi : X \dashrightarrow Y$. A projective variety X is said to be *rational* if it is birational to $\mathbb{P}^n$ for some non-negative integer n.

Definition 5.25. Two projective varieties X and Y are said to be *isomorphic* if there exists an isomorphism $\varphi : X \longrightarrow Y$.

Remark 5.26. In fact, being birational is an equivalence relation among projective varieties. This is an extremely important and profound notion in algebraic geometry. Determining which projective varieties are in the same birational equivalence class, and finding a good representative in each class, are the fundamental questions in a major branch of algebraic geometry, called *birational geometry.* As these questions are in general very difficult, a complete answer is far from being achieved. We will see some examples later.