6. Function Fields

We will study rational functions on projective varieties, and pullback of rational functions along dominant rational maps. Similar to the affine case, we will see that the field of rational functions determines the birational class of a projective variety.

6.1. **Bridge between affine and projective algebraic sets.** We have seen affine and projective algebraic sets as subsets of affine and projective spaces defined by polynomial equations. They are related in a way that is similar to affine and projective spaces. Recall that $\mathbb{P}^n$ is covered by standard affine charts U_i for $i = 0, 1, \cdots, n$.

Proposition 6.1 (From projective to affine)**.** *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and U_i a standard affine chart of $\mathbb{P}^n$. Then $X_i := X \cap U_i$ is an affine algebraic set in U_i.*

Proof. Without loss of generality, we prove the statement for $i = 0$. Assume $X = \mathbb{V}_p(f_1, \cdots, f_m)$ for some homogeneous polynomials $f_1, \cdots, f_m \in \Bbbk[z_0, \cdots, z_n]$. Then

$$\begin{aligned} p = [a_0 : \cdots : a_n] \in X \cap U_0 &\iff f_j(a_0, a_1 \cdots, a_n) = 0 \text{ for each } j \\ &\iff f_j\left(1, \frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0}\right) = 0 \text{ for each } j \\ &\iff g_j\left(\frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0}\right) = 0 \text{ for each } j \end{aligned}$$

where $g_j = f_j(1, z_1, \cdots, z_n)$. Hence $X_i = \mathbb{V}_a(g_1, \cdots, g_m)$ is an affine algebraic set. □

Remark 6.2. As in the proof, given a homogeneous polynomial (i.e. f_j), we can set one of its variables to be 1 to obtain a (not necessarily homogeneous) polynomial (i.e. g_j). This process is often called *dehomogenisation*.

Definition 6.3. Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and U_i a standard affine chart of $\mathbb{P}^n$. The affine algebraic set $X_i = X \cap U_i$ is called a *standard affine piece* of X. The decomposition $X = \cup_{i=0}^n X_i$ is called the *standard affine cover* of X.

Example 6.4. Consider the projective algebraic sets $X = \mathbb{V}_p(xy - z^2) \subseteq \mathbb{P}^2$. By setting one of the variables to be 1, we obtain the three standard affine pieces of X, which are $X_0 = \mathbb{V}_a(y - z^2) \subseteq \mathbb{A}^2$, $X_1 = \mathbb{V}_a(x - z^2) \subseteq \mathbb{A}^2$, and $X_2 = \mathbb{V}_a(xy - 1) \subseteq \mathbb{A}^2$.

We turn to another relation between affine and projective algebraic sets. Recall that $\mathbb{P}^n$ can be understood as $\mathbb{A}^n$ together with "points at infinity". We have also seen in Example 4.9 how to find points at infinity for a line in $\mathbb{A}^2$. More generally we have

Definition 6.5 (From affine to projective)**.** For any affine algebraic set $X \subseteq \mathbb{A}^n$, let $I = \mathbb{I}_a(X)$ and $\overline{I}$ be the ideal in $\Bbbk[z_0, \cdots, z_n]$ generated by the set of homogeneous polynomials

$$\left\{ z_0^{\deg f} f\left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right) \middle| f(x_1, \cdots, x_n) \in I \right\}.$$

Then the projective algebraic set $\overline{X} = \mathbb{V}_p(\overline{I})$ is called the *projective closure* of X. The points in $\{[z_0 : \cdots : z_n] \in \overline{X} \mid z_0 = 0\}$ are called *points at infinity* for X.

Remark 6.6. We have already used the above modification of a polynomial in Example 4.9; that is, first replacing all non-homogeneous coordinates by ratios of homogeneous coordinates, then clearing the denominators. This process is often called *homogenisation*. More precisely, for a polynomial $f(x_1, \cdots, x_n) \in \Bbbk[x_1, \cdots, x_n]$, assume $\deg f = d$ and let

$$f = f_0 + f_1 + \cdots + f_{d-1} + f_d$$

be its homogeneous decomposition, then the homogenisation of f is given by

$$z_0^d \cdot f\left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right) = z_0^d f_0 + z_0^{d-1} f_1 + \cdots + z_0 f_{d-1} + f_d.$$

Example 6.7. The projective closure of $\mathbb{A}^n$ is $\mathbb{P}^n$. The points at infinity are all points in H_0, namely, all points $\{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n \mid z_0 = 0\}$.

This definition is not easy to use in general, as it requires to homogenise infinitely many polynomials in $\mathbb{I}_a(X)$. The following criterion is more convenient for computations.

Proposition 6.8. *Let $X = \mathbb{V}_a(f) \subseteq \mathbb{A}^n$ be an affine hypersurface for some polynomial $f \in \Bbbk[x_1, \cdots, x_n]$ of degree d. Let*

$$\overline{f}(z_0, z_1, \cdots, z_n) = z_0^d f\left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right)$$

be the homogenisation of f. Then $\overline{X} = \mathbb{V}_p(\overline{f})$.

Proof. Non-examinable. □

Remark 6.9. In general, when an affine algebraic set X is defined by more than one polynomial, the projective closure of X is not defined by homogenisation of the generators of $\mathbb{I}_a(X)$. We will see an example in Exercise 6.3.

Example 6.10. In Example 4.9, we have seen that the projective closure of $\mathbb{V}_a(x_2 - x_1 + 1) \subseteq \mathbb{A}^2$ is $\mathbb{V}_p(x_2 - x_1 + x_0) \subseteq \mathbb{P}^2$, and that the projective closure of $\mathbb{V}_a(x_2 - x_1 - 1) \subseteq \mathbb{A}^2$ is $\mathbb{V}_p(x_2 - x_1 - x_0) \subseteq \mathbb{P}^2$. The point at infinity for both affine algebraic sets is $[0 : 1 : 1]$.

Example 6.11. We compute the projective closure and points at infinity for the heart curve $X = \mathbb{V}_a((x^2 + y^2 - 1)^3 - x^2y^3)$. We use z for the extra variable. By Proposition 6.8, the projective closure is given by one homogeneous equation of degree 6; that is

$$\overline{X} = \mathbb{V}_p((x^2 + y^2 - z^2)^3 - x^2y^3z).$$

To find the points at infinity, we set $z = 0$. Then we have $(x^2 + y^2)^3 = 0$, hence $y = \pm\sqrt{-1}x$. It follows that there are two points at infinity given by $[x : y : z] = [1 : \sqrt{-1} : 0]$ and $[1 : -\sqrt{-1} : 0]$.

Finally we briefly mention the relation of the two constructions. They are almost inverse to each other, subject to some assumptions. For simplicity, we only state the correspondece in the case of varieties. We have the following bijection. Recall that $H_0 = \mathbb{P}^n \backslash U_0$.

$$\left\{ \begin{array}{c} \text{projective varieties } X \subseteq \mathbb{P}^n \\ \text{such that } X \not\subseteq H_0 \end{array} \right\} \underset{X=\overline{Y}}{\overset{Y=X\cap U_0}{\rightleftarrows}} \left\{ \begin{array}{c} \text{affine varieties } Y \subseteq U_0 \cong \mathbb{A}^n \\ \text{such that } Y \neq \varnothing \end{array} \right\}$$

We summarise the content of the correspondence in the following result:

Proposition 6.12. *There is a bijection between projective varieties in* $\mathbb{P}^n$ *which are not contained in* $H_0 = \mathbb{P}^n \backslash U_0$ *and non-empty affine varieties in* U_0*, given by the mutually inverse correspondences of taking the standard affine piece in* U_0 *and taking the projective closure.*

Proof. Non-examinable. Interested reader can find the proof in [Section 5.5, Reid, Undergraduate Algebraic Geometry] or [Section 4.3, Fulton, Algebraic Curves]. □

The importance of the two constructions relating affine and projective varieties is that they allow us to study some properties in a relatively easier context, i.e., either affine or projective, and deduce some similar properties in the other context. We will see two examples in future lectures.