6.2. **Rational functions and function fields.** As we have seen, polynomials cannot be used to define functions on projective algebraic sets. Therefore we have to find a more flexible way to define functions on them, namely, rational functions. For simplicity, we only consider varieties. We will first define rational functions on affine varieties, then on projective varieties.

For any affine variety $X \subseteq \mathbb{A}^n$, $\mathbb{I}(X)$ is a prime ideal in $\Bbbk[x_1, \cdots, x_n]$ by Proposition 2.15. It follows that $\Bbbk[X] = \Bbbk[x_1, \cdots, x_n]/\mathbb{I}(X)$ is an integral domain by Proposition 2.12 (1).

Definition 6.13. Let $X \subseteq \mathbb{A}^n$ be an affine variety. Its *function field* $\Bbbk(X)$ is the field of fractions of the integral domain $\Bbbk[X]$. In other words,

$$\Bbbk(X) := \left\{ \frac{\varphi}{\psi} \mid \varphi, \psi \in \Bbbk[X] \text{ with } \psi \neq 0 \right\} / \sim,$$

where $\sim$ is an equivalence relation defined by

$$\frac{\varphi_1}{\psi_1} \sim \frac{\varphi_2}{\psi_2} \quad \Longleftrightarrow \quad \varphi_1\psi_2 - \psi_1\varphi_2 = 0 \in \Bbbk[X].$$

An element in $\Bbbk(X)$ is called a *rational function* on X.

Remark 6.14. Recall that φ and ψ can be given by polynomials, so we can also write

$$\Bbbk(X) = \left\{ \frac{f}{g} \mid f, g \in \Bbbk[x_1, \cdots, x_n] \text{ with } g \notin \mathbb{I}(X) \right\} / \sim,$$

where $\sim$ is an equivalence relation defined by

$$\frac{f_1}{g_1} \sim \frac{f_2}{g_2} \quad \Longleftrightarrow \quad f_1g_2 - g_1f_2 \in \mathbb{I}(X).$$

As a quick example, $\frac{1}{x}$ defines a rational function on the affine variety $X = \mathbb{A}^1$. Every polynomial function is clearly a rational function which is defined everywhere on X. But in general, a rational function is only a partially defined function on X.

Example 6.15. The coordinate ring of the affine variety $X = \mathbb{A}^n$ is $\Bbbk[\mathbb{A}^n] = \Bbbk[x_1, \cdots, x_n]$. By Definition 6.13, its function field is the field of fractions of $\Bbbk[x_1, \cdots, x_n]$, usually written as $\Bbbk(\mathbb{A}^n) = \Bbbk(x_1, \cdots, x_n)$. A rational function on $X = \mathbb{A}^n$ is given by a fraction of the form $\frac{f}{g}$ with $g \neq 0$. Two such fractions are considered to define the same rational function if and only if they can be reduced to the same form after cancelling common factors in the numerator and the denomirator.

We want to find out how to make a similar definition on projective varieties. Recall from equation (4.3) that a non-constant homogeneous polynomial cannot define a function on a projective algebraic set, because its value at a point depends on the choice of the homogeneous coordinates. However, for two homogeneous polynomials $f, g \in \Bbbk[z_0, \cdots, z_n]$

of the same degree d, their ratio $\frac{f}{g}$ is well-defined at any point $p = [a_0 : \cdots : a_n]$ provided that $g(p) \neq 0$, because for any $\lambda \neq 0$, we have

$$\frac{f(\lambda a_0, \cdots, \lambda a_n)}{g(\lambda a_0, \cdots, \lambda a_n)} = \frac{\lambda^d f(a_0, \cdots, a_n)}{\lambda^d g(a_0, \cdots, a_n)} = \frac{f(a_0, \cdots, a_n)}{g(a_0, \cdots, a_n)},$$

which is independent of the choice of the homogeneous coordinates of p. Therefore $\frac{f}{g}$ can be thought as a partially defined function on a projective variety. More precisely,

Definition 6.16. Let $X \subseteq \mathbb{P}^n$ be a projective variety. The *function field* of X is

$$\Bbbk(X) := \left\{ \frac{f}{g} \;\middle|\; f, g \in \Bbbk[z_0, \cdots, z_n] \text{ are homogeneous of the same degree, } g \notin \mathbb{I}(X) \right\} / \sim,$$

where $\sim$ is an equivalence relation defined by

$$\frac{f_1}{g_1} \sim \frac{f_2}{g_2} \iff f_1 g_2 - f_2 g_1 \in \mathbb{I}(X).$$

An element in $\Bbbk(X)$ is called a *rational function* on X.

It is in general not easy to explicitly compute the function field of a projective variety. However, the following result allows one to reduce the question to the affine situation.

Lemma 6.17. *Let $X \subseteq \mathbb{A}^n$ be an affine variety and $\overline{X} \subseteq \mathbb{P}^n$ its projective closure. Then $\Bbbk(X) \cong \Bbbk(\overline{X})$.*

Sketch of proof. (This proof is non-examinable and not covered in lectures.)

We sketch a proof. For every rational function on X

$$\frac{f(x_1, \cdots, x_n)}{g(x_1, \cdots, x_n)} \in \Bbbk(X),$$

assume $m = \max\{\deg f, \deg g\}$, then we can get a rational function on $\overline{X}$

$$\frac{z_0^m f(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0})}{z_0^m g(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0})} \in \Bbbk(\overline{X}),$$

since it is the ratio of two homogeneous polynomials of degree m. In this way we can define a map $\Bbbk(X) \to \Bbbk(\overline{X})$. On the other hand, for every rational function on $\overline{X}$

$$\frac{p(z_0, \cdots, z_n)}{q(z_0, \cdots, z_n)} \in \Bbbk(\overline{X}),$$

we have a rational function on X

$$\frac{p(1, x_1, \cdots, x_n)}{q(1, x_1, \cdots, x_n)} \in \Bbbk(X).$$

In this way we can define a map $\Bbbk(\overline{X}) \to \Bbbk(X)$. We need to verify that both maps are well-defined (i.e., independent of the choice of the representative in each equivalence class), and are homomorphisms. More work is required to check that they are inverse of each other hence are isomorphisms. □

Example 6.18. By Example 6.15 we know $\Bbbk(\mathbb{A}^n) = \Bbbk(x_1, \cdots, x_n)$. Since $\mathbb{P}^n$ is the projective closure of $\mathbb{A}^n$ by Example 6.7, we have $\Bbbk(\mathbb{P}^n) \cong \Bbbk(x_1, \cdots, x_n)$ by Lemma 6.17.

Recall that polynomial maps can pullback polynomial functions on affine algebraic sets. Similarly, a dominant rational map can pullback rational functions on projective varieties.

Definition 6.19. Let $\varphi : X \dashrightarrow Y$ be a dominant rational map between projective varieties. For every rational function g on Y, the *pullback* of g along φ is the rational function $g \circ \varphi$ on X, denoted $\varphi^*(g)$.

Example 6.20. Consider the dominant rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ studied in Example 5.18. Then the pullback of the rational function $\frac{x}{y+z} \in \Bbbk(\mathbb{P}^2)$ along φ is

$$\varphi^*\left(\frac{x}{y+z}\right) = \frac{yz}{zx+xy} \in \Bbbk(\mathbb{P}^2).$$

Recall that two affine algebraic sets are isomorphic if and only if they have isomorphic coordinate rings. A similar result holds for projective varieties.

Proposition 6.21. *A rational map $\varphi : X \dashrightarrow Y$ between projective varieties is a birational map if and only if φ is dominant and $\varphi^* : \Bbbk(Y) \longrightarrow \Bbbk(X)$ is a field isomorphism. Two projective varieties X and Y are birational if and only if $\Bbbk(X) \cong \Bbbk(Y)$.*

Proof. Non-examinable. Interested reader can find the proof in [Section 5.8, Reid, Undergraduate Algebraic Geometry] or [Section 6.6, Fulton, Algebraic Curves]. □