7. Non-singularity

The non-singularity is an algebraic version of smoothness in analysis. We will find out how to determine the non-singularity of a variety from its defining equations, and study the related notions of tangent spaces and dimensions.

7.1. **Non-singularity of irreducible hypersurfaces.** In this lecture we consider the case of irreducible hypersurfaces. We start with the affine case. Let $f \in \mathbb{k}[x_1, \cdots, x_n]$ be a non-constant irreducible polynomial. By Lemma 5.4, we know that $\mathbb{V}(f) \subseteq \mathbb{A}^n$ is an affine irreducible hypersurface.

Definition 7.1. Let $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be an affine irreducible hypersurface defined by a non-constant irreducible polynomial $f \in \mathbb{k}[x_1, \cdots, x_n]$. For any point $p \in X$, we say X is *singular* at p if $\frac{\partial f}{\partial x_i}(p) = 0$ for every i, $1 \leqslant i \leqslant n$; otherwise we say X is *non-singular* at p. If X is non-singular at every point $p \in X$, then we say X is *non-singular*; otherwise we say X is *singular*.

Remark 7.2. From Definition 7.1 we see that the singular points in $X = \mathbb{V}(f)$ form an affine algebraic set $X_{\text{sing}} = \mathbb{V}(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}) \subseteq X$. To find all singular points, we just need to solve the system of equations given by f and all its partial derivatives.

Example 7.3. Consider the affine variety $X = \mathbb{V}(f) \subseteq \mathbb{A}^2$ where $f = x^3 + y^3 - 3xy$. To find all singular points, we need to solve the system of equations given by $f = x^3 + y^3 - 3xy = 0$ and the partial derivatives $\frac{\partial f}{\partial x} = 3x^2 - 3y = 0$ and $\frac{\partial f}{\partial y} = 3y^2 - 3x = 0$. From the two partial derivatives we get $x = y^2 = x^4$, therefore $x(x^3 - 1) = 0$, which implies $x = 0$ or $x^3 = 1$. When $x = 0$, we have $y = 0$. It is clear that $(x, y) = (0, 0)$ is a solution to the system of equations. When $x^3 = 1$, we have $x^3 + y^3 - 3xy = x^3 + x^6 - 3x^3 = -1 \neq 0$. Contradition. Therefore the only point at which X is singular is $(0, 0)$.

The following result shows that $X = \mathbb{V}(f)$ cannot be singular everywhere. Recall that we always assume the underlying field $\mathbb{k}$ is an algebraically closed field of charasteristic 0.

Theorem 7.4. *Let $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be an affine hypersurface defined by a non-constant irreducible polynomial $f \in \mathbb{k}[x_1, \cdots, x_n]$. Then the set of non-singular points in X is non-empty.*

Proof. The set of singular points in X is given by

$$X_{\text{sing}} = \mathbb{V}\left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) \subseteq X.$$

Suppose on the contrary that $X_{\text{sing}} = X$, then $\frac{\partial f}{\partial x_i} \in \mathbb{I}(X)$ for every i.

Since f is an irreducible polynomial, (f) is a prime ideal by Lemma 5.4. It follows by Proposition 2.9 that $\mathbb{I}(X) = (f)$. Therefore for every i, we have

$$\frac{\partial f}{\partial x_i} = f \cdot g_i$$

for some $g_i \in \Bbbk[x_1, \cdots, x_n]$. Assume f has degree d_i in x_i. If $d_i > 0$, then $\frac{\partial f}{\partial x_i}$ has degree $d_i - 1$ in x_i, while $f \cdot g_i$ has degree at least d_i in x_i. Contradiction. Therefore $d_i = 0$. In other words, x_i does not occur in f. Since this holds for every i, f must be a constant polynomial. Contradiction. This finishes the proof of existence of non-singular points in $X = \mathbb{V}(f)$. □

Definition 7.5. Let $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be an affine irreducible hypersurface defined by a non-constant irreducible polynomial $f \in \Bbbk[x_1, \cdots, x_n]$. For any point $p = (a_1, \cdots, a_n) \in X$, the *tangent space* of X at p is the affine variety

$$T_pX := \mathbb{V}\left(\frac{\partial f}{\partial x_1}(p) \cdot (x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(p) \cdot (x_n - a_n)\right) \subseteq \mathbb{A}^n.$$

Example 7.6. Following Example 7.3, we compute the tangent spaces of X at two points $p_1 = (\frac{4}{3}, \frac{2}{3})$ and $p_2 = (0, 0)$. Recall that $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (3x^2 - 3y, 3y^2 - 3x)$. It is easy to compute that $(\frac{\partial f}{\partial x}(p_1), \frac{\partial f}{\partial y}(p_1)) = (\frac{10}{3}, -\frac{8}{3})$ and $(\frac{\partial f}{\partial x}(p_2), \frac{\partial f}{\partial y}(p_2)) = (0, 0)$. Therefore

$$\begin{aligned} T_{p_1}X &= \mathbb{V}\left(\frac{10}{3}\left(x - \frac{4}{3}\right) - \frac{8}{3}\left(y - \frac{2}{3}\right)\right) = \mathbb{V}(5x - 4y - 4), \\ T_{p_2}X &= \mathbb{V}\left(0 \cdot (x - 0) + 0 \cdot (y - 0)\right) = \mathbb{A}^2 \end{aligned}$$

are the tangent spaces of X at p_1 and p_2 respectively.

Remark 7.7. In Definition 7.5, when p is singular point of X, the defining equation of T_pX is a zero polynomial hence $T_pX = \mathbb{A}^n$, which has dimension n as a vector space over $\Bbbk$; when X is non-singular at p, the tangent space T_pX is a shift of the vector subspace $\mathbb{V}\left(\frac{\partial f}{\partial x_1}(p) \cdot x_1 + \cdots + \frac{\partial f}{\partial x_n}(p) \cdot x_n\right)$, which has dimension $n - 1$. Therefore we can say, the irreducible hypersurface $X \subseteq \mathbb{A}^n$ is non-singular at p if and only if $\dim T_pX = n - 1$; X is singular at p if and only if $\dim T_pX > n - 1$. We will generalise this conclusion to arbitrary affine varieties in next lecture.

Finally we briefly mention the case of projective irreducible hypersurfaces. Let $f \in \Bbbk[z_0, \cdots, z_n]$ be a non-constant homogeneous irreducible polynomial. By Lemma 5.4, we know that $\mathbb{V}(f) \subseteq \mathbb{P}^n$ is a projective irreducible hypersurface.

Definition 7.8. Let $X = \mathbb{V}(f) \subseteq \mathbb{P}^n$ be a projective irreducible hypersurface defined by a non-constant homogeneous irreducible polynomial $f \in \Bbbk[z_0, \cdots, z_n]$. For any point $p \in X$, we say X is *singular* at p if the affine hypersurface $X_i = X \cap U_i$ is singular at p for any standard affine piece X_i containing p; otherwise we say X is *non-singular* at p. The *tangent space* T_pX of X at p is the projective closure of T_pX_i for any standard affine piece X_i containing p. If X is non-singular at every point $p \in X$, then we say X is *non-singular*; otherwise we say X is *singular*.

Remark 7.9. A point $p \in X$ could be contained in several standard affine pieces of X. To check whether X is singular at p, and compute the tangent space of X at p, it suffices to choose one standard affine piece of X containing p. The result does not depend on the choice of the standard affine piece.

Example 7.10. Consider the projective variety $Y = \mathbb{V}_p(\overline{f}) \subseteq \mathbb{P}^2$ where $\overline{f} = x^3+y^3-3xyz$. The standard affine piece $Y \cap U_2$ is the affine variety in Examples 7.3 and 7.6. The results in the two examples imply that Y is non-singular at $p_1 = [\frac{4}{3} : \frac{2}{3} : 1] = [4 : 2 : 3]$ and singular at $p_2 = [0 : 0 : 1]$. Moreover, the tangent spaces of Y at p_1 and p_2 are given by $T_{p_1}Y = \mathbb{V}_p(5x - 4y - 4z)$ and $T_{p_2}Y = \mathbb{P}^2$.