7.2. **Non-singularity of varieties.** We generalise our discussion from last time and study non-singularity of varieties. Similarly, we first consider the case of affine varieties. for any affine variety X, we know by Corollary 1.14 that $\mathbb{I}(X)$ is finitely generated.

Definition 7.11. Let $X \subseteq \mathbb{A}^n$ be a non-empty affine variety. Assume $\mathbb{I}(X) = (f_1, \cdots, f_m)$ for some $f_1, \cdots, f_m \in \Bbbk[x_1, \cdots, x_n]$. For any point $p = (a_1, \cdots, a_n) \in X$, the *tangent space* of X at p is the affine variety

$$T_pX := \bigcap_{i=1}^{m} \mathbb{V}\left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(p) \cdot (x_j - a_j)\right) \subseteq \mathbb{A}^n.$$

Remark 7.12. We can view the tangent space T_pX as a shift of the linear subspace

$$\bigcap_{i=1}^{m} \mathbb{V}\left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(p) \cdot x_j\right) \subseteq \mathbb{A}^n$$

which is the null space of the matrix

$$M_p := \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}.$$

By the rank-nullity theorem, the dimension of T_pX is given by

$$\dim T_pX = n - \operatorname{rank} M_p.$$

Definition 7.13. Let $X \subseteq \mathbb{A}^n$ be a non-empty affine variety. The *dimension* of X is

$$\dim X = \min\{\dim T_pX \mid p \in X\}.$$

For any point $p \in X$, we say X is *singular* at p if $\dim T_pX > \dim X$; we say X is *non-singular* at p if $\dim T_pX = \dim X$. If X is non-singular at every point $p \in X$, then we say X is *non-singular*; otherwise we say X is *singular*.

Remark 7.14. By Remark 7.7, we find that Definition 7.1 for hypersurfaces is consistent with the more general Definition 7.11. We also point out: although our definition of tangent spaces and dimension involve a choice of generators in $\mathbb{I}(X)$, they are in fact independent of the choice. In other words, different choices of generators in $\mathbb{I}(X)$ always give the same tangent spaces and dimension.

Example 7.15. As a simple example, let $X = \mathbb{A}^n$, then $\mathbb{I}(X) = \{0\}$. For any point $p \in X$, it is clear that M_p is a zero matrix and $T_pX = \mathbb{A}^n$. Therefore $\dim T_pX = n - \operatorname{rank} M_p = n$. It follows that $\dim X = n$, and X is non-singular.

Example 7.16. Remark 7.7 together with Theorem 7.4 shows that $\dim X = n - 1$ for any irreducible hypersurface $X \subseteq \mathbb{A}^n$.

Example 7.17. As another simple example, let $X = \{p\} \subseteq \mathbb{A}^n$ be a single point set, where $p = (a_1, \cdots, a_n)$. By Exercise 2.3 we know $\mathbb{I}(X) = (x_1 - a_1, \cdots, x_n - a_n)$. Then we have $M_p = I_n$ is the identity matrix, and that $T_pX = \cap_{i=1}^{n} \mathbb{V}(x_i - a_i) = \{p\}$. It follows that $\dim X = 0$ and X is non-singular.

Now we consider projective varieties. Similar to the hypersurface case, the non-singularity and dimension of a projective variety can be reduced to its standard affine pieces.

Definition 7.18. Let $X \subseteq \mathbb{P}^n$ be a non-empty projective variety. The *dimension* of X is defined to be $\dim X_i$ for any non-empty standard affine piece $X_i = X \cap U_i$, denoted $\dim X$. For any point $p \in X$, we say X is *singular* at p if X_i is singular at p for any standard affine piece $X_i = X \cap U_i$ containing p; otherwise we say X is non-singular at p. If X is non-singular at every point $p \in X$, then we say X is non-singular; otherwise we say X is singular.

Remark 7.19. The dimension of a projective variety can be computed on any of its non-empty standard affine piece. Similarly whether X is singular at p can be computed on any of its standard affine piece containing p. Different standard affine pieces always give the same answer. However, in order to find all singular points in a projective variety X, we need to work with more than one standard affine piece to avoid missing any point.

A very surprising property of the dimension is its intrinsic nature.

Theorem 7.20. *Let X and Y be (affine or projective) varieties. If $\mathbb{k}(X) \cong \mathbb{k}(Y)$, then $\dim X = \dim Y$.*

Proof. Non-examinable. Interested reader can find the proof in [Sections 6.7 and 6.8, Reid, Undergraduate Algebraic Geometry] or [Section 6.5, Fulton, Algebraic Curves]. □

Remark 7.21. Theorem 7.20 shows that the dimension of a variety X only depends on its function field $\mathbb{k}(X)$. In particular, by Proposition 6.21, if two projective varieties X and Y are birational, then $\dim X = \dim Y$.

Definition 7.22. An affine (resp. projective) *algebraic curve* $C \subseteq \mathbb{A}^n$ (resp. $C \subseteq \mathbb{P}^n$) is a finite union of affine (resp. projective) varieties of dimension 1.

Finally we look at a comprehensive example.

Example 7.23. Consider the projective variety $X = \mathbb{V}_p(w+x+y+z, w^2+x^2+y^2+z^2) \subseteq \mathbb{P}^3$. We will show that X is a non-singular curve. By Definition 7.18, we need to show every standard affine piece of X is non-singular of dimension 1.

We look at the standard affine piece $X_0 = X \cap U_0 = \{p = [w : x : y : z] \in X \mid w \neq 0\}$. Then $X_0 = \mathbb{V}_a(1+x+y+z, 1+x^2+y^2+z^2) \subseteq \mathbb{A}^3$. To use Definition 7.11, we need to know that $\mathbb{I}_a(X_0) = (1+x+y+z, 1+x^2+y^2+z^2)$. This can be verified by showing the ideal $(1+x+y+z, 1+x^2+y^2+z^2)$ is prime and applying Proposition 2.9 (1). We skip the proof of this step and simply assume it is true.

For any point $p \in X_0$, we have

$$M_p = \begin{pmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \end{pmatrix}.$$

Since there are two rows in M_p and the first row is non-zero, we know that $1 \leqslant \operatorname{rank} M_p \leqslant 2$ for every point $p \in X_0$. We claim that $\operatorname{rank} M_p = 2$ for every $p \in X_0$. Otherwise, assume $\operatorname{rank} M_p = 1$ for some $p \in X_0$, then the two rows must be proportional hence $x = y = z$. However $p \in X_0$ implies that $1 + x + y + z = 0$ and $1 + x^2 + y^2 + z^2 = 0$, which become $1 + 3x = 0$ and $1 + 3x^2 = 0$. It is easy to see that they do not have common solutions. Hence such a point p does not exist. It follows that $\dim T_pX_0 = 3 - \operatorname{rank} M_p = 1$ for every $p \in X_0$. That means X_0 is non-singular, and $\dim X = \dim X_0 = 1$.

Since the defining equations of X are completely symmetric with respect to all variables, the same computation would show that all other standard affine pieces of X are non-singular. Therefore X is a non-singular curve.