8. Algebraic Curves

We study plane curves of degree up to 3.

8.1. **Lines and conics.** From now on we focus on plane curves.

Definition 8.1. A *plane curve* is a hypersurface $C = \mathbb{V}(f) \subseteq \mathbb{P}^2$ for some non-constant homogeneous polynomial $f \in \Bbbk[x, y, z]$ without repeated factors. The *degree* of C is defined to be $\deg f$. Plane curves of degrees 1, 2, 3 and 4 are called *lines*, *conics*, *cubics* and *quartics* respectively.

Example 8.2. Let $[x : y : z]$ be the homogeneous coordinates in $\mathbb{P}^2$. Every line is defined by a polynomial $f(x, y, z) = ax + by + cz$ for some $a, b, c \in \Bbbk$ which are not simultaneously zero. A line is always irreducible.

Example 8.3. Every conic is defined by a non-zero polynomial of the form $g(x, y, z) = ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2$. It is sometimes more convenient to write it in the matrix form

$$g(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We consider the factorisation of g into irreducibles. By Exercise 4.2 (1), each irreducible factor of g is also homogeneous. There are three cases:

(1) If g is an irreducible polynomial, then $\mathbb{V}(g)$ is an irreducible conic;

(2) If $g = g_1 g_2$ for coprime irreducible homogeneous polynomials g_1 and g_2 of degree 1, then $\mathbb{V}(g) = \mathbb{V}(g_1) \cup \mathbb{V}(g_2)$ is the union of two distinct lines;

(3) If $g = g_0^2$ for an irreducible homogeneous polynomial g_0 of degree 1. Since g has repeated factors, $\mathbb{V}(g)$ is not a conic. Instead, $\mathbb{V}(g) = \mathbb{V}(g_0)$ is a line. However, sometimes it is convenient to say that g defines a "double line", just to indicate that the factor g_0 is repeated.

Definition 8.4. Let $[x : y : z]$ be the homogeneous coordinates of any point in $\mathbb{P}^2$. For a fixed 3×3 invertible matrix A, define a new set of coordinates $[x' : y' : z']$ by the equation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This is called the *linear change of homogeneous coordinates* defined by A.

Remark 8.5. Why it makes sense: Multiplication of $[x : y : z]$ by any scalar $\lambda \in \Bbbk\backslash\{0\}$ results in the multiplication of $[x' : y' : z']$ by the same scalar λ, and x', y', z' cannot be all 0 unless x, y, z are all zero since A is nonsingular. So $[x' : y' : z']$ are a new system of homogeneous coordinates for points in the projective plane. Why we care: We can often reduce the defining equation of a curve to a very simple form by choosing a new system of coordinates.

Lemma 8.6. *Every line in $\mathbb{P}^2$ can be written as $\mathbb{V}(x)$ after a suitable linear change of homogeneous coordinates. A non-zero homogeneous polynomial $g(x, y, z) = ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2$ defines an irreducible conic if and only if the matrix*

$$G = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

has rank 3; g defines a union of two lines if and only if G has rank 2; g defines a double line if and only if G has rank 1. Every irreducible conic in $\mathbb{P}^2$ can be written as $\mathbb{V}(xz - y^2)$ after a suitable linear change of homogeneous coordinates.

Proof. Non-examinable. The proof follows from the Gram-Schmidt orthogonalisation in linear algebra. □

Proposition 8.7. *A line (or an irreducible conic) is isomorphic to $\mathbb{P}^1$, hence is rational.*

Proof. By Lemma 8.6, we can assume the line is $\mathbb{V}(x)$ and the conic is $\mathbb{V}(xz - y^2)$ without loss of generality. The case of a line is easy; we leave the details to the reader. The case of a conic was proved in Example 5.23. □

The following results are special cases of a famous theorem.

Theorem 8.8. *Let L be a line and D a plane curve of degree d. If L is not a component of D, then $L \cap D$ has at most d distinct points. When counting with multiplicities, L and D meet in precisely d points.*

Proof. Assume $L = \mathbb{V}(ax + by + cz)$ where a, b and c are not simultaneously zero. Without loss of generality, we can assume $c \neq 0$. Then a point $p \in L$ can be written as $p = [x : y : -\frac{a}{c}x - \frac{b}{c}y]$. Assume $D = \mathbb{V}(f)$ where $f(x, y, z)$ is a non-zero homogeneous polynomial of degree d. Then $p \in D$ if and only if $f\left(x, y, -\frac{a}{c}x - \frac{b}{c}y\right) = 0$. The left-hand side is a homogeneous polynomial of degree d in x and y. By Exercise 4.4 (2), it can be factored into a product of d homogeneous factors of degree 1 as

$$f\left(x, y, -\frac{a}{c}x - \frac{b}{c}y\right) = (r_1x + s_1y) \cdots (r_dx + s_dy) = 0.$$

Each factor $r_ix + s_iy$ determines a solution $[x : y] = [-s_i : r_i]$ which gives point $p_i = [-s_i : r_i : \frac{a}{c}s_i - \frac{b}{c}r_i] \in L \cap D$. Some of these points may be the same, so L and D meet in at most d points. When counting with the number of times each distinct point occurs as a solution, we have precisely d points. □

Remark 8.9. If $p \in L \cap D$ occurs m times as a solution, then we say L and D meet at p with *multiplicity* m. The current proof provides a systematic method to compute all intersection points of a line and a curve with multiplicities.

Remark 8.10. We briefly explain what it means by saying L is not a component of D. For example, if D is a conic, it could be the union of two lines. If L happens to be one of them, then L and D meet in more than d points, indeed, infinitely many points. The theorem indicates that if L and D meet in more than d points, then L must be a component of D.

Proposition 8.11. *Let D be an irreducible non-singular plane curve of degree $d \geqslant 2$. For any point $p \in D$, the tangent line T_pD and D meet at p with multiplicity at least 2.*

Proof. Non-examinable. But we will see some examples in exercises. □

Theorem 8.12. *Let C be a conic and D a plane curve of degree d. If C and D have no common component, then $C \cap D$ has at most $2d$ distinct points. When counting with multiplicities, C and D meet in precisely $2d$ points.*

Proof. Similar to the proof of Theorem 8.8. We leave it as an exercise. □

The more general version of the theorem is the following

Theorem 8.13 (Bézout's Theorem)**.** *Let D_1 and D_2 be plane curves of degree d_1 and d_2 respectively. Assume D_1 and D_2 have no common component, then D_1 and D_2 meet in at most d_1d_2 distinct points. When these points are counted with multiplicities, D_1 and D_2 meet in precisely d_1d_2 points.*

Proof. Non-examinable. Interested reader can find the proof in [Section 5.3, Fulton, Algebraic Curves]. □

Remark 8.14. This theorem shows that the number of intersection points of two plane curves can be read off easily from their defining equations without solving them, which is a big advantage for projective spaces. A special case of this theorem is Exercise 4.3 (2), when both plane curves have degree 1. In the other direction, this theorem can be generalised in many different ways, thus has become the starting point of a major branch of algebraic geometry, called *intersection theory*.