8.2. **Cubics.** Now we consider cubic curves. We first give a classification.

Example 8.15. Every cubic curve is defined by a non-zero homogeneous polynomial $f \in \Bbbk[x, y, z]$ of degree 3. By Exercise 4.2 (1), each irreducible factor of f is also homogeneous. There are a few cases:

(1) If f is an irreducible polynomial, then $\mathbb{V}(f)$ is an irreducible cubic;

(2) If f is the product of two irreducible factors of degree 1 and 2 respectively, then the cubic $\mathbb{V}(f) = L \cup C$ is the union of a line L and a conic C (in this case we still say $\mathbb{V}(f)$ is singular, although we have not discussed the singularity of reducible algebraic sets);

(3) If f is the product of three irreducible factors of degree 1, then $\mathbb{V}(f)$ could be the union of three distinct lines, or the union of a single line and a double line, or a triple line. The union of three distinct lines is a cubic. The other two are not.

We have seen that there is only one line and one irreducible conic up to linear changes of homogeneous coordinates. The situation is different for irreducible cubics.

Lemma 8.16. *Up to a linear change of homogeneous coordinates, every irreducible cubic curve C can be written in one of the following three forms*

(1) $C_0 = \mathbb{V}_p\left(y^2z - x(x-z)(x-\lambda z)\right)$ *for some* $\lambda \in \Bbbk \backslash \{0, 1\}$*;*

(2) $C_1 = \mathbb{V}_p\left(y^2z - x^2(x-z)\right)$*;*

(3) $C_2 = \mathbb{V}_p\left(y^2z - x^3\right)$*.*

Proof. Non-examinable. □

Remark 8.17. The defining equations in Lemma 8.16 are called the *normal forms* of irreducible cubics. By Exercise 6.2, we see that these formulas do give irreducible cubics. Moreover, by Exercise 7.3, C_0 is always non-singular; C_1 is singular at the point $[0:0:1]$, where C_1 intersects with itself; C_2 is singular at the point $[0:0:1]$, where C_2 has a corner. They are known respectively as an *non-singular cubic*, the *nodal cubic* and the *cuspidal cubic*. Each of them can be understood as the projective closure of the corresponding affine variety $\mathbb{V}_a(y^2 - x(x-1)(x-\lambda))$ or $\mathbb{V}_a(y^2 - x^2(x-1))$ or $\mathbb{V}_a(y^2 - x^3)$, with the only point at infinity $[0:1:0]$.

Proposition 8.18. *A nodal cubic curve (or a cuspidal cubic curve) is rational.*

Proof. To show a nodal cubic is rational, by Lemma 8.16, we can assume the nodal cubic is $C_1 = \mathbb{V}\left(y^2z - x^2(x-z)\right)$ without loss of generality. Consider the rational maps

$$\begin{aligned} \varphi_1 : \quad & \mathbb{P}^1 \dashrightarrow C_1; \quad [u:v] \longmapsto [u(u^2+v^2) : v(u^2+v^2) : u^3] \\ \psi_1 : \quad & C_1 \dashrightarrow \mathbb{P}^1; \quad [x:y:z] \longmapsto [x:y]. \end{aligned}$$

We will verify they are rational maps and they are inverse to each other. They are both given by homogeneous polynomials of the same degree. Moreover, φ_1 is defined, for example, at the point $[1:0]$; ψ_1 is defined, for example, at the point $[0:1:0]$. The image of ψ_1 is always in $\mathbb{P}^1$. To verify the image of φ_1 is in C, one just needs to compute

$$[v(u^2+v^2)]^2[u^3]-[u(u^2+v^2)]^2[u(u^2+v^2)-u^3]=v^2(u^2+v^2)^2u^3-u^2(u^2+v^2)^2uv^2=0.$$

Finally we show they are inverse to each other. For any point $[x:y:z]\in C_1$, we have

$$(\varphi_1\circ\psi_1)([x:y:z])=\varphi_1([x:y])=[x(x^2+y^2):y(x^2+y^2):x^3].$$

By the equation of C_1 we know $y^2z-x^2(x-z)=0$, which implies $x^3=(x^2+y^2)z$. Therefore

$$[x(x^2+y^2):y(x^2+y^2):x^3]=[x(x^2+y^2):y(x^2+y^2):z(x^2+y^2)]=[x:y:z].$$

Moreover, for any point $[u:v]\in\mathbb{P}^1$, we have

$$(\varphi_1\circ\psi_1)([u:v])=\varphi_1([u(u^2+v^2):v(u^2+v^2):u^3])=[u(u^2+v^2):v(u^2+v^2)]=[u:v].$$

This shows that C_1 is birational to $\mathbb{P}^1$, hence C_1 is rational.

To show a cuspidal cubic is rational, by Lemma 8.16, we can assume the cuspidal cubic is $C_2=\mathbb{V}(y^2z-x^3)$ without loss of generality. Consider the rational maps

$$\begin{aligned}\varphi_2:&\quad \mathbb{P}^1\dashrightarrow C_2;\quad [u:v]\longmapsto[uv^2:v^3:u^3];\\ \psi_2:&\quad C_2\dashrightarrow\mathbb{P}^1;\quad [x:y:z]\longmapsto[x:y].\end{aligned}$$

A similar proof shows C_2 is rational. We leave the details as an exercise. □

Proposition 8.19. *A non-singular cubic curve is not rational.*

Proof. Non-examinable. The idea is to show that the function field of a non-singular cubic is not isomorphic to that of $\mathbb{P}^1$. Interested reader can find the proof in [Section 2.2, Reid, Undergraduate Algebraic Geometry]. This is a fun proof. The method in the proof is called "infinite descent". There are a few famous applications of this method in the history of mathematics. It was used to prove that $\sqrt{2}$ is not a rational number, which unfortunately caused the first crisis in the foundations of mathematics. This crisis led to the discovery of irrational numbers, which was a big step forward in the development of mathematics. Another famous application of the descent method was in the proof of Fermat's last theorem. Fermat conjectured that the equation $x^m+y^m=z^m$ has no solutions in positive integers for any positive integer $m\geqslant 3$. The proof of the theorem in $m=3$ and $m=4$ cases was given by the descent method shortly after that. But it took mathematicians more than 300 years to completely solve the problem. The Andrew Wiles Building in University of Oxford was named after the British mathematician who finally proved this conjecture. □

Finally we look at some special points on a non-singular cubic.

Definition 8.20. Given a non-singular cubic curve C, a point $p \in C$ is said to be an *inflection point* of C if the tangent line T_pC meets C at p with multiplicity 3.

Remark 8.21. Recall from Proposition 8.11 that T_pC meets C at p with multiplicity at least 2. By Theorem 8.8, if p is an inflection point, then p is the only intersection point of T_pC and C; if p is not an inflection point, then T_pC and C meet at another point with multiplicity 1.

Example 8.22. We show that the point $p = [0 : 1 : 0]$ is an inflection point on the non-singular cubic $C = \mathbb{V}_p(f)$ where $f = y^2z - x^3 + xz^2$. First of all we need to find out the tangent line T_pC, which can be computed on the standard affine piece $C_1 = C \cap U_1 = \mathbb{V}_a(f_1)$ where $f_1 = z - x^3 + xz^2$. The non-homogeneous coordinates of p in U_1 is $p = (0, 0)$. Since $\frac{\partial f_1}{\partial x} = -3x^2 + z^2$ and $\frac{\partial f_1}{\partial z} = 1 + 2xz$, the tangent line $T_pC_1 = \mathbb{V}_a(0(x-0)+1(z-0)) = \mathbb{V}_a(z)$. Its projective closure is $T_pC = \mathbb{V}_p(z)$. To find the intersection points of C and T_pC, we follow the method in the proof of Theorem 8.8. A point on T_pC is given by $[x : y : 0]$. It lies in C if and only if $f(x, y, 0) = 0$, where $f(x, y, 0) = -x^3$ which has one solution $[x : y] = [0 : 1]$ with multiplicity 3. Therefore T_pC and C meet at the point $[0 : 1 : 0]$ with multiplicity 3, which proves $p = [0 : 1 : 0]$ is an inflection point on C.