10. Algebraic Surfaces

We look at a few aspects of hypersurfaces in $\mathbb{P}^3$ of low degrees.

10.1. **Planes and quadric surfaces.** From now on we focus on hypersurfaces in $\mathbb{P}^3$.

Definition 10.1. A hypersurface $S = \mathbb{V}(f) \subseteq \mathbb{P}^3$ defined by some non-constant homogeneous polynomial $f \in \mathbb{k}[z_0, z_1, z_2, z_3]$ without repeated factors is called a *surface*. The *degree* of S is defined to be $\deg f$. Surfaces of degree 1, 2, 3 and 4 are called *planes*, *quadrics*, *cubics* and *quartics* respectively.

Example 10.2. Let $[z_0 : z_1 : z_2 : z_3]$ be the homogeneous coordinates in $\mathbb{P}^3$. Every plane is defined by a polynomial $f(z_0, z_1, z_2, z_3) = a_0 z_0 + a_1 z_1 + a_2 z_2 + a_3 z_3$ for some $a_0, a_1, a_2, a_3 \in \mathbb{k}$ which are not simultaneously zero. A plane is always irreducible.

Example 10.3. Every quadric surface is defined by a non-zero homogeneous polynomial $g \in \mathbb{k}[z_0, z_1, z_2, z_3]$ of degree 2. Similar to the case of conics, it is sometimes more convenient to write it in the matrix form

$$g(z_0, z_1, z_2, z_3) = (z_0, z_1, z_2, z_3) \cdot M \cdot (z_0, z_1, z_2, z_3)^T$$

where M is a 4×4 symmetric matrix. The classification of quadric surfaces is controlled by the rank of M.

There is a notion of linear change of homogeneous coordinates in $\mathbb{P}^3$, which is literally almost the same as Definition 8.4, with all vectors having 4 components and A being a 4×4 invertible matrix.

Lemma 10.4. *Every plane in $\mathbb{P}^3$ can be written as $\mathbb{V}(z_0)$ after a suitable linear change of homogeneous coordinates. A non-zero homogeneous polynomial of degree 2*

$$g(z_0, z_1, z_2, z_3) = (z_0, z_1, z_2, z_3) \cdot M \cdot (z_0, z_1, z_2, z_3)^T$$

defines a non-singular irreducible quadric surface if and only if M has rank 4; g defines a singular irreducible quadric surface if and only if M has rank 3; g defines a union of two planes if and only if M has rank 2; g defines a double plane if and only if M has rank 1. Every non-singular quadric surface can be written as $\mathbb{V}(z_0 z_3 - z_1 z_2)$ after a suitable linear change of homogeneous coordinates.

Proof. Non-examinable. Application of Gram-Schmidt orthogonalisation again. □

Remark 10.5. A union of two planes can be thought as a singular algebraic set. A double plane is not a quadric surface. So a "non-singular quadric surface" always means a "non-singular irreducible quadric surface".

Now we turn to the rationality problem. Recall from Proposition 8.7 that a line or a non-singular conic is always isomorphic to $\mathbb{P}^1$ hence is rational. Something similar happens to surfaces.

Proposition 10.6. *A plane is isomorphic to $\mathbb{P}^2$, hence is rational. A non-singular quadric surface is birational to $\mathbb{P}^2$, hence is rational.*

Proof. By Lemma 10.4, we can assume the plane is $\mathbb{V}(z_0)$ and the non-singular quadric is $\mathbb{V}(z_0z_3 - z_1z_2)$ without loss of generality. It is easy to show that $\mathbb{V}(z_0)$ is isomorphic to $\mathbb{P}^2$; we leave the details to the reader. We have proved in Exercise 5.2 that $\mathbb{V}(z_0z_3 - z_1z_2)$ is birational to $\mathbb{P}^2$. $\square$

This result suggests that a non-singular quadric surface is not isomorphic to $\mathbb{P}^2$. Indeed, it follows from the fact that two curves in $\mathbb{P}^2$ always intersect while two curves in a quadric surface could be disjoint. The details are left as an exercise. We would like to know what precisely a quadric surface looks like. For that purpose we need the theory of multi-projective spaces. We will not discuss the theory systematically. Instead, we will only focus on this particular example and mention a few ingredients of the theory along the way. Some details in the proof are left to the reader.

Proposition 10.7. *A non-singular quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. We assume the quadric surface is $S = \mathbb{V}(z_0z_3 - z_1z_2)$. We need to find morphisms $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to S$ and $\psi : S \to \mathbb{P}^1 \times \mathbb{P}^1$, such that both compositions are identities.

The product $\mathbb{P}^1 \times \mathbb{P}^1$ is the simplest example of a *bi-projective space*. A point in it is given by a pair of points (p, q) in $\mathbb{P}^1$. If $p = [x_0 : x_1]$ and $q = [y_0 : y_1]$, then the *bi-homogeneous coordinates* of (p, q) are given by $([x_0 : x_1], [y_0 : y_1])$. Notice that for any $\lambda, \mu \in \Bbbk\backslash\{0\}$, we have $([\lambda x_0 : \lambda x_1], [\mu y_0 : \mu y_1]) = ([x_0 : x_1], [y_0 : y_1])$. We construct two morphisms:

$$\varphi : \quad \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow S; \quad ([x_0 : x_1], [y_0 : y_1]) \longrightarrow [x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1];$$

$$\psi : \quad S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1; \quad [z_0 : z_1 : z_2 : z_3] \longmapsto \begin{cases} ([z_0 : z_1], [z_0 : z_2]) & \text{if } z_0 \neq 0; \\ ([z_0 : z_1], [z_1 : z_3]) & \text{if } z_1 \neq 0; \\ ([z_2 : z_3], [z_0 : z_2]) & \text{if } z_2 \neq 0; \\ ([z_2 : z_3], [z_1 : z_3]) & \text{if } z_3 \neq 0. \end{cases}$$

We need to check they are morphisms. We have not defined the notion of a morphism in this setting, but it is very similar to a morphism between two projective varieties. All components of φ are homogeneous of the same degree with respect to the coordinates x_0 and x_1 of p, and the coordinates y_0 and y_1 of q (aka *bi-homogeneous*). All components of ψ are also homogeneous of the same degree. We observe that φ and ψ are both well-defined at every point in their domains (we leave the details to the reader). Moreover, the image of φ satisfies the defining equation of S. Hence φ is a morphism. To show ψ is a morphism, we need to verify that the image of any point in S is independent of the choice of any valid expression. More precisely, we need to verify $[z_0 : z_1] = [z_2 : z_3]$ and $[z_0 : z_2] = [z_1 : z_3]$, both of which follow from the defining equation $z_0z_3 = z_1z_2$ of S.

We check the composition $\psi \circ \varphi$ is identity. Given any point $([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$, using the first expression of ψ, we have

$$\begin{aligned}(\psi \circ \varphi)([x_0 : x_1], [y_0 : y_1]) &= \psi([x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1]) \\ &= ([x_0y_0 : x_1y_0], [x_0y_0 : x_0y_1]) \\ &= ([x_0 : x_1], [y_0 : y_1]).\end{aligned}$$

Similarly we can check that $\psi \circ \varphi$ is identity in all the other three cases.

We check the composition $\varphi \circ \psi$ is identity. Given any point $[z_0 : z_1 : z_2 : z_3] \in S$, using the first expression of ψ, we have

$$\begin{aligned}(\varphi \circ \psi)([z_0 : z_1 : z_2 : z_3]) &= \varphi([z_0 : z_1], [z_0 : z_2]) \\ &= [z_0^2 : z_0z_1 : z_0z_2 : z_1z_2] \\ &= [z_0^2 : z_0z_1 : z_0z_2 : z_0z_3] \\ &= [z_0 : z_1 : z_2 : z_3].\end{aligned}$$

Similarly we can check $\varphi \circ \psi$ is identity in all the other three cases.

To summarise, φ and ψ are mutually inverse isomorphisms. Therefore a quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Quadric surfaces are very useful in civil engineering. According to the literature, the Shukhov water tower (in Polibino, Russia, 1896, designed by Shukhov) is the first structure of this shape ever built in the world. Similar design can also be found at a few places inside and outside Sagrada Família (in Barcelona, Spain, designed by Gaudi). Nowaways numerous cooling towers in power plants are built in this shape.