10.2. **Non-singular cubic surfaces.** We have seen that non-singular cubic curves have very rich geometry. The situation is similar for cubic surfaces. The theory of cubic surfaces has a long history. It is known since 1849 that a non-singular cubic surface contains 27 lines. This discovery is one of the first results on surfaces of higher degree and is considered by many as the start of modern algebraic geometry. Many mathematicians contributed to the understanding of rich geometry of non-singular cubic surfaces. In this lecture we will take a glimpse of the theory of non-singular cubic surfaces via examples.

**Definition 10.8.** A *line* in $\mathbb{P}^3$ is a projective variety $\mathbb{V}(f,g)$, where $f,g \in \Bbbk[z_0,z_1,z_2,z_3]$ are non-zero homogeneous polynomials of degree 1 which are not proportional to each other.

*Remark* 10.9. The definition shows that a line in $\mathbb{P}^3$ is defined by the system of equations

$$\begin{cases} a_0z_0 + a_1z_1 + a_2z_2 + a_3z_3 = 0 \\ b_0z_0 + b_1z_1 + b_2z_2 + b_3z_3 = 0 \end{cases}$$

such that the coefficient matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

has rank 2. We know from linear algebra that its reduced row echelon form has two pivots, therefore the two variables corresponding to the pivots can be written as linear functions of the other variables. For example, if the pivots are in the first two columns, then

$$\begin{cases} z_0 = r_2z_2 + r_3z_3 \\ z_1 = s_2z_2 + s_3z_3 \end{cases}$$

for some $r_2, r_3, s_2, s_3 \in \Bbbk$.

**Proposition 10.10.** *The Fermat cubic surface* $S = \mathbb{V}(z_0^3 + z_1^3 + z_2^3 + z_3^3)$ *contains exactly* 27 *lines.*

*Proof.* Assume a line $L$ in $\mathbb{P}^3$ is given by $z_0 = r_2z_2 + r_3z_3$ and $z_1 = s_2z_2 + s_3z_3$ for some $r_2, r_3, s_2, s_3 \in \Bbbk$ (i.e. pivots in first two columns). Such a line lies in $S$ if and only if

$$(r_2z_2 + r_3z_3)^3 + (s_2z_2 + s_3z_3)^3 + z_2^3 + z_3^3 = 0$$

holds for all $z_2, z_3 \in \Bbbk$, hence is an identity. By comparing the coefficients, we get

$$r_2^3 + s_2^3 = -1 \tag{1}$$

$$r_3^3 + s_3^3 = -1 \tag{2}$$

$$r_2^2r_3 = -s_2^2s_3 \tag{3}$$

$$r_2r_3^2 = -s_2s_3^2 \tag{4}$$

If $r_2, r_3, s_2, s_3$ are all non-zero, then $(3)^2/(4)$ gives $r_2^3 = -s_2^3$, in contradiction to (1). Hence for a line in the cubic at least one of these numbers must be zero. By (3) $r_2$ and $r_3$ cannot be both non-zero.

If $r_2 = 0$, then by (1) $s_2^3 = -1$, hence by (3) $s_3 = 0$, which by (2) implies $r_3^3 = -1$. This gives 9 solutions $r_2 = s_3 = 0$, $s_2 = -\omega^j$, $r_3 = -\omega^k$ for $0 \leqslant j, k \leqslant 2$ and $\omega = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$ is a primitive third root of unity. We thus obtain 9 lines given by

$$z_0 + \omega^k z_3 = z_1 + \omega^j z_2 = 0, \qquad 0 \leqslant j, k \leqslant 2.$$

If $r_3 = 0$, we can similarly find out that $s_2 = 0$ and $r_2^3 = s_3^3 = -1$, hence we obtain another 9 lines given by

$$z_0 + \omega^k z_2 = z_1 + \omega^j z_3 = 0, \qquad 0 \leqslant j, k \leqslant 2.$$

As the equation of $S$ is symmetric with respect to all variables, we can allow permutations of variables to find other lines in the cubic (i.e. pivots not necessarily in first two columns). Some of the lines show up repeatedly after permutations of variables, but we get 9 new lines given by

$$z_0 + \omega^k z_1 = z_2 + \omega^j z_3 = 0, \qquad 0 \leqslant j, k \leqslant 2.$$

In summary, we have equations of all 27 lines. $\square$

**Proposition 10.11.** *The cubic surface* $S = \mathbb{V}(z_0^2 z_1 + z_1^2 z_2 + z_2^2 z_3 + z_3^2 z_0)$ *is rational.*

*Proof.* We write down two mutually inverse rational maps

$$\begin{aligned}
\varphi : &\quad S \dashrightarrow \mathbb{P}^2; \quad [z_0 : z_1 : z_2 : z_3] \longmapsto [z_0 z_3 : z_1 z_2 : z_2 z_3];\\
\psi : &\quad \mathbb{P}^2 \dashrightarrow S; \quad [r : s : t] \longmapsto [rt(rt + s^2) : -s(r^2 s + t^3) : t^2(rt + s^2) : -t(r^2 s + t^3)].
\end{aligned}$$

To check they are rational maps, we observe that they are both given by homogeneous polynomials of the same degree. It is easy to check that $\varphi([1 : -1 : 1 : -1]) = [1 : 1 : 1]$ and $\psi([1 : 1 : 1]) = [1 : -1 : 1 : -1])$, hence both $\varphi$ and $\psi$ are defined on non-empty sets. We need to show the image of $\psi$ satisfies the defining equation of $S$, which can be computed directly.

It remains to show that both $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps on the loci where they are well-defined. This is also a simple calculation. We leave the details to the reader. This shows that $S$ and $\mathbb{P}^2$ are birational. By definition, $S$ is rational. $\square$

The phenomenons in the above examples hold for every non-singular cubic surface. We summarise it in the following result.

**Theorem 10.12.** *Every non-singular cubic surface contains exactly* 27 *lines. Every non-singular cubic surface is rational.*

*Proof.* Non-examinable. Interested reader can find the proof in [Chapter 7, Reid, Undergraduate Algebraic Geometry]. $\square$

*Remark* 10.13. If we fix the degree and vary the dimension, there is major difference between non-singular cubic curves and surfaces: the former is not rational while the latter is rational. In higher dimensions, whether a cubic hypersurface is rational is a very difficult question. (There is an answer in dimension 3, but mostly unknown in dimension 4 or higher.)

Moreover, if we fix the dimension, then the number of lines in a non-singular surface depends on its degree: planes and non-singular quadric surfaces contain infinitely many lines (which we will see in an exercise); a non-singular cubic surface has 27 lines; most non-singular surfaces of higher degrees have no lines at all.

Counting special curves in various kinds of spaces turns out to be a fascinating topic in algebraic geometry, which is usually called *enumerative geometry*. These questions are not only interesting to mathematicians, but also have been extensively studied in physics, as they play an important role in string theory. The 27 lines in non-singular cubic surfaces is a first example of this type.