In all questions, $k$ is an algebraically closed field of characteristic 0.

Question 1.

(a) What is an affine algebraic set? What is an affine hypersurface? What is an affine variety? 

(b) What is a Noetherian ring? State the Hilbert basis theorem, and use it to prove that $k[x_1, \ldots, x_n]$ is a Noetherian ring for any positive integer $n$.

(c) What is the coordinate ring of an affine algebraic set $X$? Prove that the coordinate ring of the affine algebraic set $X = \mathbb{V}(x) \subseteq \mathbb{A}^2$ is isomorphic to $k[t]$.

(d) Let $X$ be an irreducible affine algebraic set. Prove that the ideal $I(X)$ is prime. (You do not need to prove $I(X)$ is an ideal.)

(e) Let $X = \mathbb{V}(y^3 - x^4) \subseteq \mathbb{A}^2$ be an affine algebraic set. Consider the polynomial map $\varphi : \mathbb{A}^1 \rightarrow X$ defined by $\varphi(t) = (t^3, t^4)$. Is $\varphi$ an isomorphism? Justify your answer. You can assume that $y^3 - x^4$ is an irreducible polynomial without proof.

Question 2.

(a) What is a homogeneous ideal? Given a homogeneous ideal $I \subseteq k[z_0, \ldots, z_n]$, what is the projective algebraic set defined by $I$? State the projective Nullstellensatz.

(b) Let $I = (f) \subseteq k[z_0, \ldots, z_n]$ for some non-constant polynomial $f$. Prove that $I$ is a prime ideal if and only if $f$ is an irreducible polynomial.

(c) What does it mean to say a rational map is dominant? Consider a morphism $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ defined by $\varphi([u : v]) = [u^2 : uv : v^2]$. Is $\varphi$ dominant? Briefly explain your reason.

(d) Let $X = \mathbb{V}(z_0 z_3 - z_1 z_2) \subseteq \mathbb{P}^3$ be a projective variety. Show that $\varphi : \mathbb{P}^2 \rightarrow X$ defined by $\varphi([u : v : w]) = [u^2 : uv : uw : vw]$ is a rational map. Show that $\varphi$ is dominant.

(e) Consider the projective algebraic set $X = \mathbb{V}(z_0 z_1 z_2, z_1 z_2 z_3, z_2 z_3 z_0, z_3 z_0 z_1) \subseteq \mathbb{P}^3$. Is $X$ a projective variety? Justify your answer. You can use the fact that the union of finitely many projective algebraic sets is still a projective algebraic set without proof.
Question 3.

(a) Let \( X \subseteq \mathbb{P}^n \) be a projective algebraic set, and \( U_0 \) a standard affine chart of \( \mathbb{P}^n \). Prove that \( X \cap U_0 \) is an affine algebraic set in \( U_0 \). \[4\]

(b) Let \( X \subseteq \mathbb{A}^n \) be an affine algebraic set. What is the projective closure of \( X \)? What are points at infinity for \( X \)? If \( X = \mathbb{V}(y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)) \subseteq \mathbb{A}^2 \), find the projective closure of \( X \) and points at infinity. \[4\]

(c) Let \( X = \mathbb{V}(f) \subseteq \mathbb{A}^n \) be an affine hypersurface defined by a non-constant irreducible polynomial \( f \in k[x_1, \cdots, x_n] \). What does it mean to say that \( X \) is singular at a point \( p \in X \)? For any point \( q = (a_1, \cdots, a_n) \in X \), what is the tangent space of \( X \) at \( q \)? \[4\]

(d) Prove that the projective variety \( X = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2 \) is non-singular. Show all your reasonings. \[4\]

(e) Find all singular points on the affine curve \( X = \mathbb{V}(f) \subseteq \mathbb{A}^2 \) where the defining polynomial \( f = (x^2 + y^2 + 1)^3 + 27x^2y^2 \). \[4\]

Question 4.

(a) What is a plane curve? What is the degree of a plane curve? Let \( C_1, C_2 \) and \( C_3 \) be irreducible non-singular plane curves of degree 1, 2 and 3 respectively. Determine whether each of them is rational. (You do not need to justify your answer for this part.) \[4\]

(b) Let \( L \) be a line and \( D \) a plane curve of degree \( d \). If \( L \) is not a component of \( D \), prove that \( L \cap D \) has at most \( d \) distinct points. Briefly explain why, when counting with multiplicities, \( L \) and \( D \) meet in precisely \( d \) points. \[4\]

(c) Show that the nodal cubic curve \( C = \mathbb{V}(y^2z - x^2(x - z)) \subseteq \mathbb{P}^2 \) is rational. \[4\]

(d) Consider the non-singular cubic curve \( C = \mathbb{V}(y^2z - x^3 - 4xz^2) \subseteq \mathbb{P}^2 \). Let \( O = [0 : 1 : 0] \) be the identity element in the group law. Find the order of the subgroup generated by the point \( P = [2 : 4 : 1] \in C \). \[4\]

(e) Consider the non-singular cubic curve \( C = \mathbb{V}(x^3 + y^3 + z^3) \subseteq \mathbb{P}^2 \). Let \( O = [1 : -1 : 0] \) be the identity element in the group law. Consider the point \( P = [0 : 1 : -1] \in C \). Find \( -P \) in the group law. \[4\]