MA40188 ALGEBRAIC CURVES 2015/16 SEMESTER 1
MOCK EXAM

In all questions, $\mathbb{k}$ is an algebraically closed field of characteristic 0.

Question 1.

(a) What is an affine algebraic set? What is an affine hypersurface? What is an affine variety? [4]

(b) What is a Noetherian ring? State the Hilbert basis theorem, and use it to prove that $\mathbb{k}[x_1, \cdots, x_n]$ is a Noetherian ring for any positive integer n. [4]

(c) What is the coordinate ring of an affine algebraic set X? Prove that the coordinate ring of the affine algebraic set $X = \mathbb{V}(x) \subseteq \mathbb{A}^2$ is isomorphic to $\mathbb{k}[t]$. [4]

(d) Let X be an irreducible affine algebraic set. Prove that the ideal $\mathbb{I}(X)$ is prime. (You do no need to prove $\mathbb{I}(X)$ is an ideal.) [4]

(e) Let $X = \mathbb{V}(y^3 - x^4) \subseteq \mathbb{A}^2$ be an affine algebraic set. Consider the polynomial map $\varphi : \mathbb{A}^1 \to X$ defined by $\varphi(t) = (t^3, t^4)$. Is φ an isomorphism? Justify your answer. You can assume that $y^3 - x^4$ is an irreducible polynomial without proof. [4]

Question 2.

(a) What is a homogeneous ideal? Given a homogeneous ideal $I \subseteq \mathbb{k}[z_0, \cdots, z_n]$, what is the projective algebraic set defined by I? State the projective Nullstellensatz. [4]

(b) Let $I = (f) \subseteq \mathbb{k}[z_0, \cdots, z_n]$ for some non-constant polynomial f. Prove that I is a prime ideal if and only if f is an irreducible polynomial. [4]

(c) What does it mean to say a rational map is dominant? Consider a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ defined by $\varphi([u : v]) = [u^2 : uv : v^2]$. Is φ dominant? Briefly explain your reason. [4]

(d) Let $X = \mathbb{V}(z_0z_3 - z_1z_2) \subseteq \mathbb{P}^3$ be a projective variety. Show that $\varphi : \mathbb{P}^2 \dashrightarrow X$ defined by $\varphi([u : v : w]) = [u^2 : uv : uw : vw]$ is a rational map. Show that φ is dominant. [4]

(e) Consider the projective algebraic set $X = \mathbb{V}(z_0z_1z_2, z_1z_2z_3, z_2z_3z_0, z_3z_0z_1) \subseteq \mathbb{P}^3$. Is X a projective variety? Justify your answer. You can use the fact that the union of finitely many projective algebraic sets is still a projective algebraic set without proof. [4]

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Question 3.

(a) Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and U_0 a standard affine chart of $\mathbb{P}^n$. Prove that $X \cap U_0$ is an affine algebraic set in U_0. [4]

(b) Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. What is the projective closure of X? What are points at infinity for X? If $X = \mathbb{V}(y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)) \subseteq \mathbb{A}^2$, find the projective closure of X and points at infinity. [4]

(c) Let $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be an affine hypersurface defined by a non-constant irreducible polynomial $f \in \mathbb{k}[x_1, \cdots, x_n]$. What does it mean to say that X is singular at a point $p \in X$? For any point $q = (a_1, \cdots, a_n) \in X$, what is the tangent space of X at q? [4]

(d) Prove that the projective variety $X = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^2$ is non-singular. Show all your reasonings. [4]

(e) Find all singular points on the affine curve $X = \mathbb{V}(f) \subseteq \mathbb{A}^2$ where the defining polynomial $f = (x^2 + y^2 + 1)^3 + 27x^2y^2$. [4]

Question 4.

(a) What is a plane curve? What is the degree of a plane curve? Let C_1, C_2 and C_3 be irreducible non-singular plane curves of degree 1, 2 and 3 respectively. Determine whether each of them is rational. (You do not need to justify your answer for this part.) [4]

(b) Let L be a line and D a plane curve of degree d. If L is not a component of D, prove that $L \cap D$ has at most d dictinct points. Briefly explain why, when counting with multiplicities, L and D meet in precisely d points. [4]

(c) Show that the nodal cubic curve $C = \mathbb{V}(y^2z - x^2(x - z)) \subseteq \mathbb{P}^2$ is rational. [4]

(d) Consider the non-singular cubic curve $C = \mathbb{V}(y^2z - x^3 - 4xz^2) \subseteq \mathbb{P}^2$. Let $O = [0 : 1 : 0]$ be the identity element in the group law. Find the order of the subgroup generated by the point $P = [2 : 4 : 1] \in C$. [4]

(e) Consider the non-singular cubic curve $C = \mathbb{V}(x^3 + y^3 + z^3) \subseteq \mathbb{P}^2$. Let $O = [1 : -1 : 0]$ be the identity element in the group law. Consider the point $P = [0 : 1 : -1] \in C$. Find $-P$ in the group law. [4]