MA40188 ALGEBRAIC CURVES 2015/16 SEMESTER 1
MOCK EXAM BRIEF SOLUTIONS

Question 1.

(a) See Definitions 1.2, 1.3, 2.13.

(b) See Definition 1.12, Theorem 1.13, Corollary 1.14.

(c) See Definition 3.3, Example 3.5.

(d) See Proposition 2.15.

(e) This part is almost identical to Exercise 3.3 (4). We claim φ is not an isomorphism. Since $y^3 - x^4$ is irreducible, the ideal $(y^3 - x^4) \subseteq \Bbbk[x, y]$ is a prime ideal hence radical. So $\mathbb{I}(X) = (y^3 - x^4)$ and $\Bbbk[X] = \Bbbk[x, y]/(y^3 - x^4)$.

The polynomial map $\varphi : \mathbb{A}^1 \to X$ induces a $\Bbbk$-algebra homomorphism $\varphi^* : \Bbbk[X] \to \Bbbk[\mathbb{A}^1]$, or equivalently, $\varphi^* : \Bbbk[x, y]/(y^3 - x^4) \to \Bbbk[t]$, such that $\varphi^*(x) = t^3$ and $\varphi^*(y) = t^4$. For an arbitrary $f(x, y) \in \Bbbk[x, y]$, we have $\varphi^*(f) = f(t^3, t^4)$. We observe that every term in $\varphi^*(f)$ is either a constant term or a term of degree at least 3, hence φ^* is not surjective; for instance, t and t^2 are not in the image of φ^*. Since φ^* is not an isomorphism, φ is not an isomorphism of affine algebraic sets.

Question 2.

(a) See Definitions 4.13, 4.16, Theorem 4.24.

(b) See Lemma 5.4.

(c) See Definition 5.14, Example 5.15.

(d) See Exercise 5.2 (2).

(e) We claim that X is the collection of all points $[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3$ such that at least two coordinates are zero. On one hand, for any point with at least two zero coordinates, all the four polynomials $z_0z_1z_2, z_1z_2z_3, z_2z_3z_0, z_3z_0z_1$ achieve zero because each of them has a zero factor. On the other hand, for any point with at most one zero coordinate, it has at least three non-zero coordinates. Without loss of generality, we assume z_1, z_2, z_3 are non-zero, then $z_1z_2z_3$ achieves a non-zero value, hence the point is not in X.

Date: December 6, 2015.

Now let $L_{ij} = \mathbb{V}(z_i, z_j) \subseteq \mathbb{P}^3$ be the projective algebraic set of all points whose z_i and z_j coordinates are zero. Then we have $X = L_{01} \cup L_{02} \cup L_{03} \cup L_{12} \cup L_{13} \cup L_{23}$. Let $X_1 = L_{01} \cup L_{02} \cup L_{03}$ and $X_2 = L_{12} \cup L_{13} \cup L_{23}$, then X_1 and X_2 are projective algebraic sets contained in X. They are strictly smaller than X because, e.g., $[1:1:0:0] \in X \backslash X_1$ and $[0:0:1:1] \in X \backslash X_2$. It follows that X is reducible, hence not a projective variety.

Question 3.

(a) See Proposition 6.1.

(b) See Definition 6.5, Exercise 6.2 (2).

(c) See Definitions 7.1, 7.5.

(d) See Exercise 7.2 (1)(2). Notice that y and z are switched.

(e) We have $\frac{\partial f}{\partial x} = 3(x^2+y^2+1)^2 \cdot 2x + 54xy^2$ and $\frac{\partial f}{\partial y} = 3(x^2+y^2+1)^2 \cdot 2y + 54x^2y$.

Case 1. If $x = 0$, then $f = 0$ implies $(y^2+1)^3 = 0$, hence $y^2 = -1$. Then $(x, y) = (0, \pm\sqrt{-1})$. It is easy to check that for these values we do have $f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, hence they are singular points.

Case 2. If $y = 0$, we can similarly get singular points $(x, y) = (\pm\sqrt{-1}, 0)$.

Case 3. If neither x nor y is zero, then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ implies $(x^2+y^2+1)^2 = -9y^2 = -9x^2$, hence $x^2 = y^2$ and $(2x^2+1)^2 = -9x^2$. Moreover $f = 0$ implies $(2x^2+1)^3 = -27x^4$. Then we have $-9^3x^6 = (2x^2+1)^6 = 27^2x^8$. Notice that $9^3 = 3^6 = 27^2$, hence we have $-x^6 = x^8$. Since $x \neq 0$, we have $x^2 = -1$. But then $(2x^2+1)^2 = 1 \neq 9 = -9x^2$. Contradiction. So there is no singular point in this case.

To summarise, there are four singular points on X, which are $(x, y) = (0, \pm\sqrt{-1})$ and $(\pm\sqrt{-1}, 0)$.

Question 4.

(a) See Definition 8.1. C_1 and C_2 are rational. C_3 is not rational.

(b) See Theorem 8.8.

(c) See Proposition 8.18.

(d) See Exercise 9.2 (2).

(e) We follow the third paragraph in the proof of Proposition 9.8.

Step 1. We look for the third intersection point $\overline{O}$ of the line $T_O C$ and the curve C. Consider the standard affine chart $C_0 = \mathbb{V}(f_0) \subseteq \mathbb{A}^2$ where $f_0 = 1 + y^3 + z^3$. The point $O = (-1, 0) \in C_0$. Since $\frac{\partial f_0}{\partial y} = 3y^2$ and $\frac{\partial f_0}{\partial z} = 3z^2$, we have $\frac{\partial f_0}{\partial y}(O) = 3$ and $\frac{\partial f_0}{\partial z}(O) = 0$, hence $T_O C_0 = \mathbb{V}(3(y+1)) = \mathbb{V}(y+1) \subseteq \mathbb{A}^2$. It follows that $T_O C = \mathbb{V}(x+y) \subseteq \mathbb{P}^2$. For any point $q = [x : y : z] \in T_O C$, we have $y = -x$. If $q \in C$, we then have $x^3 + (-x)^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution $[x : z] = [1 : 0]$ with multiplicity 3. This means $T_O C$ meet C at one point $[1 : -1 : 0]$ with multiplicity 3, hence $\overline{O} = [1 : -1 : 0] = O$.

Step 2. We look for the third intersection point $-P$ of the line $\overline{O}P$ and the curve C. The line $\overline{O}P$ is given by

$$\det \begin{pmatrix} x & 1 & 0 \\ y & -1 & 1 \\ z & 0 & -1 \end{pmatrix} = 0;$$

or equivalently $x + y + z = 0$. We write $x = -y - z$, then the defining polynomial of C becomes $x^3 + y^3 + z^3 = (-y - z)^3 + y^3 + z^3 = 0$; or equivalently $-3y^2 z - 3yz^2 = 0$. It can be factored into $-3yz(y+z) = 0$. Hence the three solutions are $[y : z] = [0 : 1]$, $[1 : 0]$ and $[1 : -1]$. It follows the three intersection points are $[x : y : z] = [-1 : 0 : 1]$, $[-1 : 1 : 0]$ and $[0 : 1 : -1]$. Since $\overline{O} = [1 : -1 : 0] = [-1 : 1 : 0]$ and $P = [0 : 1 : -1]$, we conclude that $-P = [-1 : 0 : 1]$.