Solution 1.1. *Examples of algebraic sets.* There are many possible answers.

1. One possible answer is \( X = \mathbb{V}(x(x - 1), y(y - 1)) \).
2. One possible answer is \( X = \mathbb{V}(x(y - 1), y(x - 1)) \).
3. One possible answer is \( X = \mathbb{V}(xy, y(y - 1)) \).
4. This algebraic set is the union of the three coordinate axes. In other words, it is the set of points \((x, y, z) \in \mathbb{A}^3\) with at least two zero coordinates.

Solution 1.2. *Prove Proposition 1.7.*

1. Given any \( p \in \mathbb{V}(S_1) \), we have \( f(p) = 0 \) for every \( f \in S_1 \). Since every \( g \in S_2 \) is also an element in \( S_1 \), we have \( g(p) = 0 \). Hence \( p \in \mathbb{V}(S_2) \).
2. We have that \( \emptyset = \mathbb{V}(1) \) and \( \mathbb{A}^n = \mathbb{V}(0) \).
3. We first prove \( \bigcap_{\alpha} (\mathbb{V}(S_\alpha)) \subseteq \mathbb{V}(\bigcup_{\alpha} S_\alpha) \). Given any point \( p \in \bigcap_{\alpha} (\mathbb{V}(S_\alpha)) \), we have \( p \in \mathbb{V}(S_\alpha) \) for every \( \alpha \). Then for every \( f \in \bigcup_{\alpha} S_\alpha \), there exists some \( \alpha_0 \) such that \( f \in S_{\alpha_0} \), therefore \( f(p) = 0 \) since \( p \in \mathbb{V}(S_{\alpha_0}) \). This shows that \( p \in \mathbb{V}(\bigcup_{\alpha} S_\alpha) \).

We then prove \( \bigcap_{\alpha} (\mathbb{V}(S_\alpha)) \supseteq \mathbb{V}(\bigcup_{\alpha} S_\alpha) \). Given any point \( q \in \mathbb{V}(\bigcup_{\alpha} S_\alpha) \), we have \( g(p) = 0 \) for every \( g \in \bigcup_{\alpha} S_\alpha \). In particular, for every \( \alpha \), we have \( p \in \mathbb{V}(S_\alpha) \). Therefore \( p \in \cap_{\alpha} (\mathbb{V}(S_\alpha)) \).
4. We first prove \( \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \subseteq \mathbb{V}(S) \). Given any \( p \in \mathbb{V}(S_1) \), we have \( f(p) = 0 \) for every \( f \in S_1 \). Therefore for every \( f \in S \) with \( f \in S_1 \) and \( g \in S_2 \), \( (fg)(p) = f(p)g(p) = 0 \). Hence \( p \in \mathbb{V}(S) \). This proves \( \mathbb{V}(S_1) \subseteq \mathbb{V}(S) \). Similarly we have \( \mathbb{V}(S_2) \subseteq \mathbb{V}(S) \). Therefore \( \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \subseteq \mathbb{V}(S) \).

We then prove \( \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \supseteq \mathbb{V}(S) \). For every \( p \in \mathbb{V}(S) \), we need to show that \( p \in \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \). If not, then \( p \notin \mathbb{V}(S_1) \) and \( p \notin \mathbb{V}(S_2) \). This means there exists some \( f_0 \in S_1 \) and \( g_0 \in S_2 \), such that \( f_0(p) \neq 0 \) and \( g_0(p) \neq 0 \). It follows that \( (f_0g_0)(p) = f_0(p)g_0(p) \neq 0 \). Since \( f_0g_0 \in S \), this implies \( p \notin \mathbb{V}(S) \). Contradiction. This proves \( \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \supseteq \mathbb{V}(S) \).

We then use induction to prove that \( \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_n) \) is an algebraic set for every positive integer \( n \). When \( n = 1 \), \( \mathbb{V}(S_1) \) is by definition an algebraic set. Assume the statement holds for \( n = k \), then \( \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_k) \) is an algebraic set, say, \( \mathbb{V}(S') \). When \( n = k + 1 \), we can write

\[
\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_k) \cup \mathbb{V}(S_{k+1})
= (\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_k)) \cup \mathbb{V}(S_{k+1})
= \mathbb{V}(S') \cup \mathbb{V}(S_{k+1})
\]

which is still an algebraic set by the statement we just proved.
Solution 1.3. Examples of algebraic sets.

(1) We know that $\mathbb{A}^1$ and $\emptyset$ are algebraic sets by Proposition 1.7 (2). For any non-empty finite subset of $\mathbb{A}^1$, say, $X = \{c_1, c_2, \ldots, c_k\}$, we have $X = \mathbb{V}((x - c_1)(x - c_2)\cdots(x - c_k))$, hence is an algebraic set.

(2) Say $X = \mathbb{V}(S)$ is an algebraic set in $\mathbb{A}^1$. If $S$ does not contain any non-zero polynomial, then $X = \mathbb{A}^1$. Otherwise, there is some $f(x) \in S$ which is a non-zero polynomial. Every point in $X$ must be a root of $f(x)$, hence $X$ is a subset of the all roots of $f(x)$. Since $f(x)$ has only finitely many roots, $X$ has at most finitely many elements.

(3) There are many possible counterexamples and here is one of them: for every positive integer $n$, let $X_n = \{n\}$ be a single-point set. Then $X_n$ is an algebraic set. But their union $\cup_n X_n$ is the set of all positive integers, which is an infinite set, hence is not an algebraic set by part (2).

Solution 1.4. Prove Proposition 1.16.

(1) We check that $q^{-1}(J) = \{r \in R \mid r + I \in J\}$ is an ideal in $R$. For any $a_1, a_2 \in q^{-1}(J)$, we have $a_1 + I, a_2 + I \in J$ hence $(a_1 + a_2) + I = (a_1 + I) + (a_2 + I) \in J$, which implies $a_1 + a_2 \in q^{-1}(J)$. On the other hand, for any $r \in R$ and $a \in q^{-1}(J)$, we have $a + I \in J$ hence $ra + I = (r + I)(a + I) \in J$ hence $ra \in q^{-1}(J)$. Therefore $q^{-1}(J)$ is an ideal in $R$.

(2) For every $a \in q^{-1}(J_1)$, we have $a + I \in J_1$. Since $J_1 \subseteq J_2$, we have $a + I \in J_2$. Hence $a \in q^{-1}(J_2)$. This verifies that $q^{-1}(J_1) \subseteq q^{-1}(J_2)$.

(3) Suppose $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ is an ascending chain of ideals in $R/I$. Then by parts (1) and (2) we have $q^{-1}(J_1) \subseteq q^{-1}(J_2) \subseteq q^{-1}(J_3) \subseteq \cdots$ is an ascending chain of ideals in $R$. Since $R$ is a Noetherian ring, this chain stabilises by Proposition 1.15. That means, there exists some positive integer $N$, such that $q^{-1}(J_i) = q^{-1}(J_N)$ for every $i \geq N$. In other words, $q^{-1}(J_i)$ and $q^{-1}(J_N)$ contain precisely the same cosets of $I$ in $R$. Therefore $J_i = J_N$ for every $i \geq N$.

(4) We showed in part (3) that every ascending chain of ideals in $R/I$ stabilises. Therefore $R/I$ is a Noetherian ring by Proposition 1.15.