Solutions to Exercise Sheet 1

Solution 1.1. *Examples of algebraic sets.* There are many possible answers.

(1) One possible answer is $X = \mathbb{V}(x(x-1), y(y-1))$.

(2) One possible answer is $X = \mathbb{V}(x(y-1), y(x-1))$.

(3) One possible answer is $X = \mathbb{V}(xy, y(y-1))$.

(4) This algebraic set is the union of the three coordinate axes. In other words, it is the set of points $(x, y, z) \in \mathbb{A}^3$ with at least two zero coordinates.

Solution 1.2. *Prove Proposition 1.7.*

(1) Given any $p \in \mathbb{V}(S_1)$, we have $f(p) = 0$ for every $f \in S_1$. Since every $g \in S_2$ is also an element in S_1, we have $g(p) = 0$. Hence $p \in \mathbb{V}(S_2)$.

(2) We have that $\varnothing = \mathbb{V}(1)$ and $\mathbb{A}^n = \mathbb{V}(0)$.

(3) We first prove $\cap_\alpha(\mathbb{V}(S_\alpha)) \subseteq \mathbb{V}(\cup_\alpha S_\alpha)$. Given any point $p \in \cap_\alpha(\mathbb{V}(S_\alpha))$, we have $p \in \mathbb{V}(S_\alpha)$ for every α. Then for every $f \in \cup_\alpha S_\alpha$, there exists some α_0 such that $f \in S_{\alpha_0}$, therefore $f(p) = 0$ since $p \in \mathbb{V}(S_{\alpha_0})$. This shows that $p \in \mathbb{V}(\cup_\alpha S_\alpha)$.

We then prove $\cap_\alpha(\mathbb{V}(S_\alpha)) \supseteq \mathbb{V}(\cup_\alpha S_\alpha)$. Given any point $q \in \mathbb{V}(\cup_\alpha S_\alpha)$, we have $g(p) = 0$ for every $g \in \cup_\alpha S_\alpha$. In particular, for every α, we have $p \in \mathbb{V}(S_\alpha)$. Therefore $p \in \cap_\alpha(\mathbb{V}(S_\alpha))$.

(4) We first prove $\big(\mathbb{V}(S_1) \cup \mathbb{V}(S_2)\big) \subseteq \mathbb{V}(S)$. Given any $p \in \mathbb{V}(S_1)$, we have $f(p) = 0$ for every $f \in S_1$. Therefore for every $fg \in S$ with $f \in S_1$ and $g \in S_2$, $(fg)(p) = f(p)g(p) = 0$. Hence $p \in \mathbb{V}(S)$. This proves $\mathbb{V}(S_1) \subseteq \mathbb{V}(S)$. Similarly we have $\mathbb{V}(S_2) \subseteq \mathbb{V}(S)$. Therefore $\big(\mathbb{V}(S_1) \cup \mathbb{V}(S_2)\big) \subseteq \mathbb{V}(S)$.

We then prove $\big(\mathbb{V}(S_1) \cup \mathbb{V}(S_2)\big) \supseteq \mathbb{V}(S)$. For every $p \in \mathbb{V}(S)$, we need to show that $p \in \mathbb{V}(S_1) \cup \mathbb{V}(S_2)$. If not, then $p \notin \mathbb{V}(S_1)$ and $p \notin \mathbb{V}(S_2)$. This means there exists some $f_0 \in S_1$ and $g_0 \in S_2$, such that $f_0(p) \neq 0$ and $g_0(p) \neq 0$. It follows that $(f_0 g_0)(p) = f_0(p)g_0(p) \neq 0$. Since $f_0 g_0 \in S$, this implies $p \notin \mathbb{V}(S)$. Contradiction. This proves $\big(\mathbb{V}(S_1) \cup \mathbb{V}(S_2)\big) \supseteq \mathbb{V}(S)$.

We then use induction to prove that $\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_n)$ is an algebraic set for every positive integer n. When $n = 1$, $\mathbb{V}(S_1)$ is by definition an algebraic set. Assume the statement holds for $n = k$, then $\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_k)$ is an algebraic set, say, $\mathbb{V}(S')$. When $n = k + 1$, we can write

$$\begin{aligned} & \mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_k) \cup \mathbb{V}(S_{k+1}) \\ = & \big(\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \cup \cdots \cup \mathbb{V}(S_k)\big) \cup \mathbb{V}(S_{k+1}) \\ = & \mathbb{V}(S') \cup \mathbb{V}(S_{k+1}) \end{aligned}$$

which is still an algebraic set by the statement we just proved.

Solution 1.3. *Examples of algebraic sets.*

(1) We know that $\mathbb{A}^1$ and $\varnothing$ are algebraic sets by Proposition 1.7 (2). For any non-empty finite subset of $\mathbb{A}^1$, say, $X = \{c_1, c_2, \cdots, c_k\}$, we have $X = \mathbb{V}((x - c_1)(x - c_2)\cdots(x - c_k))$, hence is an algebraic set.

(2) Say $X = \mathbb{V}(S)$ is an algebraic set in $\mathbb{A}^1$. If S does not contain any non-zero polynomial, then $X = \mathbb{A}^1$. Otherwise, there is some $f(x) \in S$ which is a non-zero polynomial. Every point in X must be a root of $f(x)$, hence X is a subset of the all roots of $f(x)$. Since $f(x)$ has only finitely many roots, X has at most finitely many elements.

(3) There are many possible counterexamples and here is one of them: for every positive integer n, let $X_n = \{n\}$ be a single-point set. Then X_n is an algebraic set. But their union $\cup_n X_n$ is the set of all positive integers, which is an infinite set, hence is not an algebraic set by part (2).

Solution 1.4. *Prove Proposition 1.16.*

(1) We check that $q^{-1}(J) = \{r \in R \mid r + I \in J\}$ is an ideal in R. For any $a_1, a_2 \in q^{-1}(J)$, we have $a_1 + I, a_2 + I \in J$ hence $(a_1 + a_2) + I = (a_1 + I) + (a_2 + I) \in J$, which implies $a_1 + a_2 \in q^{-1}(J)$. On the other hand, for any $r \in R$ and $a \in q^{-1}(J)$, we have $a + I \in J$ hence $ra + I = (r + I)(a + I) \in J$ hence $ra \in q^{-1}(J)$. Therefore $q^{-1}(J)$ is an ideal in R.

(2) For every $a \in q^{-1}(J_1)$, we have $a + I \in J_1$. Since $J_1 \subseteq J_2$, we have $a + I \in J_2$. Hence $a \in q^{-1}(J_2)$. This verifies that $q^{-1}(J_1) \subseteq q^{-1}(J_2)$.

(3) Suppose $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ is an ascending chain of ideals in R/I. Then by parts (1) and (2) we have $q^{-1}(J_1) \subseteq q^{-1}(J_2) \subseteq q^{-1}(J_3) \subseteq \cdots$ is an ascending chain of ideals in R. Since R is a Noetherian ring, this chain stablises by Proposition 1.15. That means, there exists some positive integer N, such that $q^{-1}(J_i) = q^{-1}(J_N)$ for every $i \geqslant N$. In other words, $q^{-1}(J_i)$ and $q^{-1}(J_N)$ contain precisely the same cosets of I in R. Therefore $J_i = J_N$ for every $i \geqslant N$.

(4) We showed in part (3) that every ascending chain of ideals in R/I stabilises. Therefore R/I is a Noetherian ring by Proposition 1.15.