Solutions to Exercise Sheet 3

Solution 3.1. *Example: the graph of a polynomial function.*

(1) Every component of π is given by a polynomial, and the image of any point in $\mathbb{A}^n$ is clearly in $\mathbb{A}^r$, so π is a polynomial map.

(2) Since X is an algebraic set, we can write $X = \mathbb{V}(S)$ where S is a set of polynomials in $x_1, \cdots, x_n$. Each polynomial in S can also be thought as a polynomial in $x_1, \cdots, x_n, x_{n+1}$. Assume the polynoial function f is represented by a polynomial $F \in \Bbbk[x_1, \cdots, x_n]$. Then consider the set of polynomials $T = S \cup \{x_{n+1} - F\} \subseteq \Bbbk[x_1, \cdots, x_n, x_{n+1}]$. We claim $G(f) = \mathbb{V}(T)$.

To prove the claim, we need to show mutual inclusions. Given any point $p = (a_1, \cdots, a_n, a_{n+1}) \in G(f)$, we have $(a_1, \cdots, a_n) \in X$ and $a_{n+1} = f(a_1, \cdots, a_n)$. The former implies that p is a solution to all polynomials in S, and the latter implies that p is a solution to the polynomial $x_{n+1} - F$. It follows that $p \in \mathbb{V}(T)$.

Given any point $q = (a_1, \cdots, a_n, a_{n+1}) \in \mathbb{V}(T)$, since x_{n+1} does not occur in any polynomial in S, we know that $(a_1, \cdots, a_n) \in \mathbb{V}(S)$. Moreover $a_{n+1} - F(a_1, \cdots, a_n) = 0$ implies that $a_{n+1} = F(a_1, \cdots, a_n) = f(a_1, \cdots, a_n)$. Hence $q \in G(f)$. This finishes the proof of the claim $G(f) = \mathbb{V}(T)$, which implies $G(f)$ is an algebraic set.

(3) The first n components of φ are obviously polynomials in $a_1, \cdots, a_n$. Since f is a polynomial map, it can also be represented by a polynomial $F \in \Bbbk[x_1, \cdots, x_n]$. It remains to check the image of φ is always in $G(f)$, which is clear from the definition of $G(f)$.

(4) We define $\psi : G(f) \to X$ as the projection map to the first n components. Namely, $\psi(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_n)$. It is clearly a polynomial map. We compute both compositions. Given any $p = (a_1, \cdots, a_n) \in X$, we have

$$(\psi \circ \varphi)(p) = \psi(a_1, \cdots, a_n, f(a_1, \cdots, a_n)) = (a_1, \cdots, a_n) = p.$$

Given any $q = (a_1, \cdots, a_n, a_{n+1}) \in G(f)$, we have

$$(\varphi \circ \psi)(q) = \varphi(a_1, \cdots, a_n) = (a_1, \cdots, a_n, f(a_1, \cdots, a_n)) = (a_1, \cdots, a_n, a_{n+1}) = q.$$

Therefore φ (hence also ψ) is an isomorphism.

(5) Let $X = \mathbb{A}^1$, and $f(x) = x^2 \in \Bbbk[x]$, then part (4) recovers Example 3.14.

Solution 3.2. *Example: a nodal cubic.*

(1) Both components in φ are polynomials in t. Since

$$\begin{aligned} y^2 - x^3 - x^2 &= (t^3 - t)^2 - (t^2 - 1)^3 - (t^2 - 1)^2 \\ &= t^6 - 2t^4 + t^2 - t^6 + 3t^4 - 3t^2 + 1 - t^4 + 2t^2 - 1 = 0, \end{aligned}$$

we conclude $\varphi(t) \in X$ for every $t \in \mathbb{A}^1$. Hence φ is a polynomial map.

(2) To show φ is surjective but not injective on points, take any point $q = (x, y) \in X$. There are two cases. If $x = 0$, then by the defining equation of X we also have $y = 0$. It is easy to see that the point $q = (0, 0)$ is the image of the point $t = 1$ or $t = -1$. Hence φ is not injective on points. If $x \neq 0$, then consider $t = \frac{y}{x}$. To find its image, notice that

$$t^2 - 1 = \frac{y^2}{x^2} - 1 = \frac{y^2 - x^2}{x^2} = \frac{x^3}{x^2} = x;$$
$$t^3 - t = t \cdot (t^2 - 1) = \frac{y}{x} \cdot x = y.$$

Therefore $\varphi(t) = (x, y)$, which means the point $q = (x, y)$ is in the image of φ. The two cases together show that φ is surjective on points. Since we have proved φ is not injective on points, it cannot be an isomorphism by Remark 3.15.

(3) Use contradiction. Assume $y^2 - x^3 - x^2 = f(x, y)g(x, y)$ for non-constant polynomials $f, g \in \Bbbk[x, y]$. Since the left-hand side has degree 2 in y, the degrees of f and g in y must be either 2 and 0, or 1 and 1. In the first case we can write

$$y^2 - x^3 - x^2 = (y^2 f_2(x) + y f_1(x) + f_0(x)) \cdot g(x).$$

Comparing coefficients of y^2 we find $f_2(x)g(x) = 1$, hence $g(x)$ must be a constant. Contradiction. In the second case we can write

$$y^2 - x^3 - x^2 = (y f_1(x) + f_0(x)) \cdot (y g_1(x) + g_0(x)).$$

Comparing coefficients of y^2 we find $f_1(x)g_1(x) = 1$. Without loss of generality we can assume $f_1(x) = g_1(x) = 1$. Comparing coefficients of y we find $f_0(x) + g_0(x) = 0$. Comparing constant terms we find $-x^3 - x^2 = f_0(x)g_0(x) = -f_0(x)^2$, hence $f_0(x)^2 = x^3 + x^2$, which is also a contradiction since $x^3 + x^2 = x^2(x + 1)$ is not a square. So we conclude that $y^2 - x^3 - x^2$ is irreducible. By Exercise 2.2 (1) we know $I = (y^2 - x^3 - x^2)$ is a prime ideal. By Proposition 2.12 (2) we know I is a radical ideal. By Proposition 2.9 (1) we know $\mathbb{I}(X) = I$. By Proposition 2.15 we know X is an irreducible algebraic set, i.e. an affine variety.

Solution 3.3. *Example: a cuspidal cubic.*

(1) Both components in φ are polynomials in t. Since

$$y^2 - x^3 = (t^3)^2 - (t^2)^3 = 0,$$

we conclude $\varphi(t) \in X$ for every $t \in \mathbb{A}^1$. Hence φ is a polynomial map.

(2) To show φ is injective and surjective on points, take any point $q = (x, y) \in X$. There are two cases. If $x = 0$, then by the defining equation of X we also have $y = 0$. Assume $\varphi(t) = (0, 0)$, then there is a unique point $t = 0$ whose image is $(0, 0)$. If $x \neq 0$, assume $\varphi(t) = (x, y)$, then we must have $t = \frac{y}{x}$, so there is at most

one point whose image is (x, y). To check that its image is indeed (x, y), notice that

$$t^2 = \frac{y^2}{x^2} = \frac{x^3}{x^2}x;$$
$$t^3 = t \cdot t^2 = \frac{y}{x} \cdot x = y.$$

Therefore $\varphi(t) = (x, y)$, which means there is a unique point $t \in \mathbb{A}^1$ whose image is the point $q = (x, y)$. The two cases together show that φ is injective and surjective on points.

(3) Use contradiction. Assume $y^2 - x^3 = f(x, y)g(x, y)$ for non-constant polynomials $f, g \in \Bbbk[x, y]$. Since the left-hand side has degree 2 in y, the degrees of f and g in y must be either 2 and 0, or 1 and 1. In the first case we can write

$$y^2 - x^3 = (y^2 f_2(x) + y f_1(x) + f_0(x)) \cdot g(x).$$

Comparing coefficients of y^2 we find $f_2(x)g(x) = 1$, hence $g(x)$ must be a constant. Contradiction. In the second case we can write

$$y^2 - x^3 = (y f_1(x) + f_0(x)) \cdot (y g_1(x) + g_0(x)).$$

Comparing coefficients of y^2 we find $f_1(x)g_1(x) = 1$. Without loss of generality we can assume $f_1(x) = g_1(x) = 1$. Comparing coefficients of y we find $f_0(x) + g_0(x) = 0$. Comparing constant terms we find $-x^3 = f_0(x)g_0(x) = -f_0(x)^2$, hence $f_0(x)^2 = x^3$, which is also a contradiction since x^3 is not a square. So we conclude that $y^2 - x^3$ is irreducible. By Exercise 2.2 (1) we know $I = (y^2 - x^3)$ is a prime ideal. By Proposition 2.12 (2) we know I is a radical ideal. By Proposition 2.9 (1) we know $\mathbb{I}(X) = I$. By Proposition 2.15 we know X is an irreducible algebraic set, i.e. an affine variety.

(4) By part (3) we have $\Bbbk[X] = \Bbbk[x, y]/(y^2 - x^3)$. To write down the pullback map explicitly, we notice that $\varphi^*(x) = t^2$ and $\varphi^*(y) = t^3$. Therefore for any polynomial map on X represented by a polynomial $f(x, y) \in \Bbbk[x, y]$, its image $\varphi^*(f) = f(t^2, t^3)$; that means, we simply replace every occurence of x by t^2 and y by t^3. It is clear that $\varphi^*(f)$ is a polynomial in t. We claim that it has no term of degree 1 in t. Indeed, the image of the constant term of f is still the same constant, and the image of any other monomial of f is a monomial in t of degree at least 2. This claim implies that φ^* is not surjective, because any polynomial in t with a non-zero degree 1 term is not in the image of φ^*. In particular, $t \in \Bbbk[t] = \Bbbk[\mathbb{A}^1]$ is not in the image of φ^*. Hence φ^* is not an isomorphism. By Proposition 3.22, φ is not an isomorphism.

Solution 3.4. *Example: the twisted cubic, revisited.*

(1) We define the polynomial map $\psi : X \to \mathbb{A}^1$ by $\psi(x, y, z) = x$. It is clearly a polynomial map as its only component is a polynomial. For any $t \in \mathbb{A}^1$, we have $(\psi \circ \varphi)(t) = \psi(t, t^2, t^3) = t$. For any $(x, y, z) \in X$, we have $(\varphi \circ \psi)(x, y, z) = \varphi(x) = (x, x^2, x^3) = (x, y, z)$. This shows that φ is an isomorphism.

(2) We first write down the pullback map φ^* explicitly. By Exercise 2.4 (3), we have $\Bbbk[X] = \Bbbk[x, y, z]/\mathbb{I}(X) = \Bbbk[x, y, z]/(y - x^2, z - x^3)$. We also have $\Bbbk[\mathbb{A}^1] = \Bbbk[t]$. The pullback of the coordinate functions are given by $\varphi^*(x) = t$, $\varphi^*(y) = t^2$ and $\varphi^*(z) = t^3$. Therefore φ^* is given by

$$\varphi^* : \quad \Bbbk[x, y, z]/(y - x^2, z - x^3) \longrightarrow \Bbbk[t]; \quad f(x, y, z) \longmapsto f(t, t^2, t^3).$$

We actually have proved in Exercise 2.4 (2) that φ^* is an isomorphism. Indeed, φ^* is surjective because every $p(t) \in \Bbbk[t]$ is the image of $p(x) \in \Bbbk[x, y, z]$ (or rather, the coset $p(x) + \mathbb{I}(X)$ in the quotient ring). Moreover, φ^* is injective because if the image of $f(x, y, z)$ is the zero polynomial in $\Bbbk[t]$, it must be in $\mathbb{I}(X)$, which means that the only element in the kernel of φ^* is the coset $0 + \mathbb{I}(X)$, which is the zero element in the quotient ring. Therefore by Proposition 3.22, we conclude that φ is an isomorphism.