Solutions to Exercise Sheet 4

Solution 4.1. *Get familiar with projective spaces.*

(1) Since there is only one 1-dimensional linear subspace in $\mathbb{A}^1$, $\mathbb{P}^0$ is a point. $\mathbb{P}^1 = U_0 \cup H_0$ where $U_0 \cong \mathbb{A}^1$ is an affine space and $H_0 \cong \mathbb{P}^0$ is a point. Therefore $\mathbb{P}^1$ has just one more point than $\mathbb{A}^1$. When $\Bbbk = \mathbb{C}$, $U_0 \cong \mathbb{A}^1_{\mathbb{C}} = \mathbb{C}^1$ is the complex plane. To view $\mathbb{P}^1$ as a bubble, imagine we remove a point from the surface of a bubble (or a globe), the remaining part can be stretched into the complex plane.

A point $p \in \mathbb{P}^n$ belongs to only one of the standard affine chart U_i if and only if p has only one non-zero homogeneous coordinate. We can assume this non-zero homogeneous coordinate to be 1, otherwise we can divide all components by it. So the point p can be given by $p = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ with 1 at a certain position and 0 at all other positions. There are $n + 1$ such points.

(2) We regard x_1 and x_2 as non-homogeneous coordinates and substitute $x_1 = \frac{z_1}{z_0}$ and $x_2 = \frac{z_2}{z_0}$. The equation $x_2^2 - x_1^2 - 1 = 0$ becomes $\frac{z_2^2}{z_0^2} - \frac{z_1^2}{z_0^2} - 1 = 0$. We clear the denominators to allow z_0 to be zero, then we get $z_2^2 - z_1^2 - z_0^2 = 0$. To find the points at infinity, set $z_0 = 0$, then we have $z_2^2 - z_1^2 = 0$, hence $z_2 = \pm z_1$. As points in $\mathbb{P}^2$ we get two solutions $[z_0 : z_1 : z_2] = [0 : 1 : 1]$ or $[0 : 1 : -1]$, which are the points at infinity for $\mathbb{V}_a(x_2^2 - x_1^2 - 1)$. This example tells us that a hyperbola has two "asymptotic directions", which is easy to understand since a hyperbola has two asymptotes.

For $\mathbb{V}_a(x_2^2 - x_1^2)$, we still substitute $x_1 = \frac{z_1}{z_0}$ and $x_2 = \frac{z_2}{z_0}$. The equation $x_2^2 - x_1^2 = 0$ becomes $\frac{z_2^2}{z_0^2} - \frac{z_1^2}{z_0^2} = 0$. We clear the denominators to allow z_0 to be zero, then we get $z_2^2 - z_1^2 = 0$. To find the points at infinity, set $z_0 = 0$, then we still have $z_2^2 - z_1^2 = 0$, hence $z_2 = \pm z_1$. As points in $\mathbb{P}^2$ we get two solutions $[z_0 : z_1 : z_2] = [0 : 1 : 1]$ or $[0 : 1 : -1]$, which are the points at infinity for $\mathbb{V}_a(x_2^2 - x_1^2)$. The result is not surprising, because the polynomial $x_2^2 - x_1^2$ defines precisely the two asymptotes of the hyperbola in the previous case.

For $\mathbb{V}_a(x_2^2 - x_1^3)$, we still substitute $x_1 = \frac{z_1}{z_0}$ and $x_2 = \frac{z_2}{z_0}$. The equation $x_2^2 - x_1^3 = 0$ becomes $\frac{z_2^2}{z_0^2} - \frac{z_1^3}{z_0^3} = 0$. We clear the denominators to allow z_0 to be zero, then we get $z_0 z_2^2 - z_1^3 = 0$. To find the points at infinity, set $z_0 = 0$, then we get $-z_1^3 = 0$, hence $z_1 = 0$. As points in $\mathbb{P}^2$ we get one solution $[z_0 : z_1 : z_2] = [0 : 0 : 1]$, which is the point at infinity for $\mathbb{V}_a(x_2^2 - x_1^3)$.

Solution 4.2. *Properties of homogeneous polynomials and ideals.*

(1) We write the homogeneous decompositions of g and h as

$$g = g_M + g_{M-1} + \cdots + g_{m+1} + g_m,$$
$$h = h_N + h_{N-1} + \cdots + h_{n+1} + h_n,$$

where M and m are the maximal and minimal degrees of non-zero monomials in g respectively; similarly N and n are the maximal and minimal degrees of non-zero monomials in h respectively. Then the degree of every monomial in the product $f = gh$ is between $m + n$ and $M + N$. Moreover, the sum of all degree $M + N$ monomials in f is given by $g_M h_N$, which is non-zero since both g_M and h_N are non-zero. Similarly, the sum of all degree $m+n$ monomials in f is given by $g_m h_n$, which is non-zero since both g_m and h_n are non-zero. If f is homogeneous, we must have $M + N = m + n$, which is only possible when $M = m$ and $N = n$. Therefore both g and h are homogeneous.

(2) Assume I is a homogeneous ideal. Since $\mathbb{k}[z_0, \cdots, z_n]$ is a Noetherian ring, I is finitely generated. So we can write $I = (f_1, \cdots, f_m)$ for some $f_1, \cdots, f_m \in I$ which are not necessarily homogeneous polynomials. However, each f_i has a homogeneous decomposition, say, $f_i = f_{i,0} + f_{i,1} + \cdots + f_{i,d_i}$ where d_i is the degree of f_i. We claim that I is generated by all the $f_{i,j}$'s; that is,

$$I = (f_{1,0}, \cdots, f_{1,d_1}, f_{2,0}, \cdots, f_{2,d_2}, \cdots\cdots, f_{m,0}, \cdots f_{m,d_m}).$$

On one hand, since I is a homogeneous ideal, each $f_{i,j} \in I$, which proves "$\supseteq$". On the other hand, we notice that every element $h \in I$ can be written as $h = f_1 g_1 + \cdots + f_m g_m$ for some $g_1, \cdots, g_m \in \mathbb{k}[z_0, \cdots, z_n]$, which can be expanded as $h = f_{1,0} g_1 + \cdots + f_{1,d_1} g_1 + \cdots\cdots + f_{m,0} g_m + \cdots + f_{m,d_m} g_m$, which proves "$\subseteq$". The claim shows that I can be generated by finitely many homogeneous polynomials.

Conversely, assume $I = (p_1, \cdots, p_l)$ for finitely many homogeneous polynomials $p_1, \cdots, p_l \in \mathbb{k}[z_0, \cdots, z_n]$, with $\deg p_i = e_i$. Given any polynomial $q \in I$, assume the homogeneous decomposition of q is $q = q_0 + \cdots + q_k$, where k is the degree of q. We need to show that every $q_j \in I$. Since $q \in I$, we can write $q = p_1 r_1 + \cdots + p_l r_l$ for some $r_1, \cdots, r_l \in \mathbb{k}[z_0, \cdots, z_n]$. For each j with $0 \leqslant j \leqslant k$, by comparing the degree j terms we get $q_j = p_1 r_{1,j-e_1} + \cdots + p_l r_{l,j-e_l}$, where each $r_{i,j-e_i}$ is the sum of all degree $j - e_i$ monomials in r_i. Since $I = (p_1, \cdots, p_l)$, we conclude that $q_j \in I$ for every j, which implies I is a homogeneous ideal.

(3) Given any point $p \in \mathbb{V}(I)$, we have $g(p) = 0$ for every homogeneous polynomial $g \in I$. In particular, $f_i(p) = 0$ for every i. Therefore $p \in \mathbb{V}(S)$. This proves $\mathbb{V}(I) \subseteq \mathbb{V}(S)$.

On the other hand, given any point $q \in \mathbb{V}(S)$, we have $f_i(q) = 0$ for every i. For any homogeneous polynomial $g \in I$, we can write $g = f_1 g_1 + \cdots + f_m g_m$ for some $g_1, \cdots, g_m \in \mathbb{k}[z_0, \cdots, z_n]$. Then $g(q) = f_1(q) g_1(q) + \cdots + f_m(q) g_m(q) = 0$. (Rigorously speaking, one should argue that each g_i can be chosen to be a homogeneous polynomial of degree equal to $\deg g - \deg f_i$, which can be achieved by replacing each g_i with its homogeneous part of degree equal to $\deg g - \deg f_i$.) This proves that $\mathbb{V}(S) \subseteq \mathbb{V}(I)$.

Solution 4.3. *Projective spaces are better than affine spaces!*

(1) Let the two points be $p = [p_0 : p_1 : p_2]$ and $q = [q_0 : q_1 : q_2]$. A line $\mathbb{V}(a_0z_0 + a_1z_1 + a_2z_2)$ passes through these two points if and only if the following system of linear equations in a_0, a_1, a_2 hold

$$p_0a_0 + p_1a_1 + p_2a_2 = 0,$$
$$q_0a_0 + q_1a_1 + q_2a_2 = 0.$$

Since p and q are distinct points in $\mathbb{P}^2$, the two rows in the coefficient matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$$

are linearly independent, hence the matrix has rank 2. By the theorem of rank-nullity, the solution space to the system has dimension 1. Let $\mathbf{v} = (a_0, a_1, a_2)$ be a non-zero solution, then every solution can be written as $\lambda\mathbf{v}$ for some $\lambda \in \Bbbk$. The solution $\mathbf{v}$ defines a line $\mathbb{V}(a_0z_0 + a_1z_1 + a_2z_2)$ through the points p and q. It remains to show the uniqueness. When $\lambda = 0$, we have $\lambda\mathbf{v} = (0, 0, 0)$ which does not define a line. For every $\lambda \in \Bbbk\backslash\{0\}$, the line $\mathbb{V}(\lambda a_0z_0 + \lambda a_1z_1 + \lambda a_2z_2)$ is the same as $\mathbb{V}(a_0z_0 + a_1z_1 + a_2z_2)$. Therefore the line through p and q is unique.

(2) Let the two lines by $\mathbb{V}(a_0z_0 + a_1z_1 + a_2z_2)$ and $\mathbb{V}(b_0z_0 + b_1z_1 + b_2z_2)$. A point $[z_0 : z_1 : z_2]$ lies on both lines if and only if it is a solution of the following system of linear equations in z_0, z_1, z_2

$$a_0z_0 + a_1z_1 + a_2z_2 = 0,$$
$$b_0z_0 + b_1z_1 + b_2z_2 = 0.$$

Since the two lines are distinct, the two rows in the coefficient matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

are linearly independent, hence the matrix has rank 2. By the theorem of rank-nullity, the solution space to the system has dimension 1. Let $\mathbf{w} = (z_0, z_1, z_2)$ be a non-zero solution, then every solution can be written as $\lambda\mathbf{w}$ for some $\lambda \in \Bbbk$. The solution $\mathbf{w}$ defines a point $[z_0 : z_1 : z_2]$ of intersection. It remains to show the uniqueness. When $\lambda = 0$, we have $\lambda\mathbf{w} = (0, 0, 0)$ which does not define a point in $\mathbb{P}^2$. For every $\lambda \in \Bbbk\backslash\{0\}$, the point $[\lambda z_0 : \lambda z_1 : \lambda z_2]$ is the same as the point $[z_0 : z_1 : z_2]$. Therefore the two lines meet at a unique point in $\mathbb{P}^2$.

Solution 4.4. *Example of projective algebraic sets.*

(1) The empty set $\varnothing = \mathbb{V}(1)$ and the entire $\mathbb{P}^1 = \mathbb{V}(0)$. For any non-empty finite subset of $\mathbb{P}^1$, say $\{[u_1 : v_1], [u_2 : v_2], \cdots, [u_k : v_k]\}$, it can be written as $\mathbb{V}(f)$ for a homogeneous polynomial $f = (v_1z_0 - u_1z_1)(v_2z_0 - u_2z_1)\cdots(v_kz_0 - u_kz_1) \in \Bbbk[z_0, z_1]$. Therefore every set stated in the question is a projective algebraic set in $\mathbb{P}^1$.

(2) Let $f \in \Bbbk[z_0, z_1]$ be a homogeneous polynomial of degree d. Assume z_0^e be the highest power of z_0 dividing f for some $e \leqslant d$. Then we can write

$$\begin{aligned} f &= c_0z_0^d + c_1z_0^{d-1}z_1 + \cdots + c_{d-e}z_0^ez_1^{d-e} \\ &= z_0^d \cdot \left(c_0 + c_1\frac{z_1}{z_0} + \cdots + c_{d-e}\frac{z_1^{d-e}}{z_0^{d-e}}\right). \end{aligned}$$

We consider the polynomial $g(x) = c_0 + c_1x + \cdots + c_{d-e}x^{d-e}$. If g is constant, then $f = c_0z_0^d$ is a product of d homogeneous polynomials of degree 1. If g is not a constant, then it can be factored into a product of polynomials of degree 1 as $g(x) = (a_1 + b_1x)\cdots(a_{d-e} + b_{d-e}x)$. Then we have

$$\begin{aligned} f &= z_0^d \cdot \left(a_1 + b_1 \cdot \frac{z_1}{z_0}\right) \cdots \left(a_{d-e} + b_{d-e} \cdot \frac{z_1}{z_0}\right) \\ &= z_0^e \cdot (a_1z_0 + b_1z_1)\cdots(a_{d-e}z_0 + b_{d-e}z_1) \end{aligned}$$

which is also a product of d homogeneous polynomials of degree 1.

(3) Let $X \subseteq \mathbb{P}^1$ be a projective algebraic set. By Corollary 4.18, we assume $X = \mathbb{V}(S)$ for a finite set S of homogeneous polynomials in $\Bbbk[z_0, z_1]$. If S does not have any non-zero polynomial then $X = \mathbb{P}^1$. Otherwise, assume $f \in S$ is a non-zero homogeneous polynomial of degree d. By part (2) we can write $f = (a_1z_0 + b_1z_1)\cdots(a_dz_0 + b_dz_1)$ (each factor z_0 can be written as $1 \cdot z_0 + 0 \cdot z_1$). For every $p = [u : v] \in X$, we have $f(p) = 0$, hence a certain factor of f vanishes at p; more precisely, $a_iu + b_iv = 0$ for some i. Therefore $p = [b_i : -a_i]$. There are at most d points of this kind, hence X contains only finitely many points.