Solutions to Exercise Sheet 5

Solution 5.1. *Example: linear embedding and linear projection.*

(1) All components are given by homogeneous polynomials of degree 1. For every point $[z_0 : z_1] \in \mathbb{P}^1$, we have either $z_0 \neq 0$ or $z_1 \neq 0$, hence $\varphi([z_0 : z_1]) = [z_0 : z_1 : 0 : 0]$ has at least one non-zero coordinate, hence is clearly a point in $\mathbb{P}^3$. Therefore φ is a morphism. It is not dominant, because for the projective algebraic set $W = \mathbb{V}(z_2, z_3) \subseteq \mathbb{P}^3$, we have $\varphi([z_0 : z_1]) \in W$ for every point $[z_0 : z_1] \in \mathbb{P}^1$.

(2) All components are given by homogeneous polynomials of degree 1. The map is not defined at every point in $\mathbb{P}^3$, but for every point $[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3$ with $z_2 \neq 0$ or $z_3 \neq 0$, its image $\psi([z_0 : z_1 : z_2 : z_3]) = [z_2 : z_3]$ has at least one non-zero coordinate, and is clearly a point in $\mathbb{P}^1$. Therefore ψ is a rational map. To see it is dominant, we first claim that ψ is surjective. In fact, for every point $[z_2 : z_3] \in \mathbb{P}^1$, we have that $[z_2 : z_3] = \psi([z_0 : z_1 : z_2 : z_3])$ for any choice of $z_0, z_1 \in \mathbb{k}$. Since ψ is surjective, we can apply Lemma 5.16 and choose $Z = \varnothing$ to conclude that ψ is dominant.

(3) The composition is not well-defined because for every $[z_0 : z_1] \in \mathbb{P}^1$, we have $(\psi \circ \varphi)([z_0 : z_1]) = \psi([z_0 : z_1 : 0 : 0]) = [0 : 0]$ which is not a point in $\mathbb{P}^1$. This shows that $\psi \circ \varphi$ is nowhere well-defined, which violates the second condition in the definition of a rational map.

Solution 5.2. *Example: the cooling tower.*

(1) Assume we can write $y_0y_3 - y_1y_2 = fg$ for some $f, g \in \mathbb{k}[y_0, y_1, y_2, y_3]$. Since the polynomial $y_0y_3 - y_1y_2$ has degree 1 in y_0, the degrees of f and g in y_0 should be 0 and 1 respectively. Without loss of generality we assume $f = y_0f_1 + f_0$ and $g = g_0$, where $f_1, f_0, g_0 \in \mathbb{k}[y_1, y_2, y_3]$. By comparing the coefficients of terms of degree 1 and 0 in y_0, we get $f_1g_0 = y_3$ and $f_0g_0 = -y_1y_2$. Therefore g_0 is a common factor of y_3 and $-y_1y_2$, which has to be a constant. This implies g is a constant, hence $y_0y_3 - y_1y_2$ is irreducible. Since it is a homogeneous polynomial, $\mathbb{V}(y_0y_3 - y_1y_2)$ is a projective algebraic set. By Lemma 5.4, the principal ideal $I = (y_0y_3 - y_1y_2)$ in $\mathbb{k}[y_0, y_1, y_2, y_3]$ is a prime ideal. Hence by Lemma 4.17, $\mathbb{V}(y_0y_3 - y_1y_2) = \mathbb{V}(I)$, which is a projective variety by Proposition 5.2.

(2) It is clear that all components of φ are given by homogeneous polynomials of degree 2. For any point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$, if x_0 is non-zero, or x_1 and x_2 are simultaneously non-zero, the image $\varphi(p)$ has at least one non-zero component. Hence φ is defined on a non-empty subset of $\mathbb{P}^2$. To show its image is always in Y, we find that $y_0y_3 - y_1y_2 = (x_0^2)(x_1x_2) - (x_0x_1)(x_0x_2) = 0$. Therefore φ is a rational map.

To show that φ is dominant, we observe that every point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ with $y_0 \neq 0$ is the image of the point $p = [y_0 : y_1 : y_2]$. Indeed, $\varphi(p) = [y_0^2 : y_0y_1 : y_0y_2 : y_1y_2] = [y_0^2 : y_0y_1 : y_0y_2 : y_0y_3] = [y_0 : y_1 : y_2 : y_3] = q$. Set $Z = \mathbb{V}(y_0y_3 - y_1y_2, y_0)$, then $Z \subseteq Y$, and is strictly smaller than Y (e.g. $[1 : 0 : 0 : 0] \in Y \backslash Z$). And every point $q \in Y \backslash Z$ is in the image of φ. By Lemma 5.16, φ is dominant.

(3) We first realise that every component of ψ is a homogeneous polynomial of degree 1. ψ is well-defined at every point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ such that y_0, y_1, y_2 are not simultaneously zero (e.g. $[1 : 0 : 0 : 0]$ is such a point). Hence it is defined on a non-empty subset of Y. The image $\psi(q)$ is always a point in $\mathbb{P}^2$ if it is defined. Therefore ψ is a rational map.

To show ψ is dominant, we first observe that each point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ with $x_0 \neq 0$ is the image of the point $q = [x_0 : x_1 : x_2 : \frac{x_1x_2}{x_0}]$. Indeed, q is a well-defined point since $x_0 \neq 0$, and $q \in Y$ since it satisfies the defining equation of Y. The expression that defines ψ gives $\psi(q) = p$. If we set $Z = \mathbb{V}(x_0)$, then $Z \subsetneq \mathbb{P}^2$. Since every point in $\mathbb{P}^2 \backslash Z$ is in the image of ψ, we conclude that ψ is dominant by Lemma 5.16.

(4) We show that φ and ψ are mutually inverse rational maps. For every point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ at which $\psi \circ \varphi$ is defined, we have $(\psi \circ \varphi)(p) = \psi([x_0^2 : x_0x_1 : x_0x_2 : x_1x_2]) = [x_0^2 : x_0x_1 : x_0x_2] = [x_0 : x_1 : x_2] = p$. For every point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ at which $\varphi \circ \psi$ is defined, we have $(\varphi \circ \psi)(q) = \varphi([y_0 : y_1 : y_2]) = [y_0^2 : y_0y_1 : y_0y_2 : y_1y_2] = [y_0^2 : y_0y_1 : y_0y_2 : y_0y_3] = [y_0 : y_1 : y_2 : y_3] = q$. Therefore φ and ψ are mutually inverse birational maps. It follows that Y is birational to $\mathbb{P}^2$, hence Y is rational.

Solution 5.3. *Example: the projective twisted cubic.*

(1) All components of φ are homogeneous of the same degree 3. For every point $[u : v] \in \mathbb{P}^1$, we have either $u \neq 0$ or $v \neq 0$, therefore either $u^3 \neq 0$ or $v^3 \neq 0$, hence $\varphi([u : v]) = [u^3 : u^2v : uv^2 : v^3]$ is always a well-defined point. To show that $\varphi([u : v]) \in Y$, we need to check all defining polynomial of Y are satisfied. Indeed, we have

$$y_0y_2 - y_1^2 = (u^3)(uv^2) - (u^2v)^2 = 0;$$
$$y_1y_3 - y_2^2 = (u^2v)(v^3) - (uv^2)^2 = 0;$$
$$y_0y_3 - y_1y_2 = (u^3)(v^3) - (u^2v)(uv^2) = 0.$$

We conclude that φ is a morphism.

(2) We define $\psi : Y \longrightarrow \mathbb{P}^1$ in the following way: for every point $[y_0 : y_1 : y_2 : y_3] \in Y$, let $\psi([y_0 : y_1 : y_2 : y_3]) = [y_0 : y_1]$ or $[y_2 : y_3]$. We first check that ψ is a morphism.

Both expressions used to define ψ are given by homogeneous polynomials of degree 1. For any point $[y_0 : y_1 : y_2 : y_3]$, if either y_0 or y_1 is non-zero (e.g. $[1 : 0 : 0 : 0]$), then the first expression applies; if either y_2 or y_3 is non-zero (e.g. $[0 : 0 : 0 : 1]$), then the second expression applies. This shows that both expressions are defined on non-empty subsets of Y. Moreover, for any point $[y_0 : y_1 : y_2 : y_3]$, at least one of its coordinates is non-zero, hence at least one of the expressions can be used to compute its image under ψ, hence ψ is defined at every point in Y. The image $\psi(q)$ for any point $q \in Y$ is clearly a point in $\mathbb{P}^1$.

To show ψ is a morphism, it remains to show that, if the two expressions are both defined at a certain point $q = [y_0 : y_1 : y_2 : y_3] \in Y$, then they give the same image. For such a point q, we claim $y_0 \neq 0$; otherwise $y_1^2 = y_0y_2 = 0$, which implies the first expression is invalid. Similarly, we claim $y_3 \neq 0$; otherwise $y_2^2 = y_1y_3 = 0$, which implies the second expression is invalid. Therefore $y_1y_2 = y_0y_3 \neq 0$, which implies $y_1 \neq 0$ and $y_2 \neq 0$. So all coordinates of q are non-zero. For such a point q, we have $[y_0 : y_1] = [y_0y_3 : y_1y_3] = [y_1y_2 : y_1y_3] = [y_2 : y_3]$, hence both expressions give the same image of q.

Finally we check that φ and ψ are mutually inverse to each other. Given any point $p = [u : v] \in \mathbb{P}^1$, we have

$$(\psi \circ \varphi)(p) = \psi([u^3 : u^2v : uv^2 : v^3]) = \begin{cases} [u^3 : u^2v] = [u : v]; \\ [uv^2 : v^3] = [u : v]. \end{cases}$$

For any point $q = [y_0 : y_1 : y_2 : y_3] \in Y$, we notice that $y_0y_1^2 = y_0 \cdot y_0y_2 = y_0^2y_2$ and $y_1^3 = y_1 \cdot y_0y_2 = y_0 \cdot y_1y_2 = y_0 \cdot y_0y_3 = y_0^2y_3$. Therefore if we use the first expression that defines ψ, we have

$$\begin{aligned} (\varphi \circ \psi)(q) &= \varphi([y_0 : y_1]) \\ &= [y_0^3 : y_0^2y_1 : y_0y_1^2 : y_1^3] \\ &= [y_0^3 : y_0^2y_1 : y_0^2y_2 : y_0^2y_3] \\ &= [y_0 : y_1 : y_2 : y_3]. \end{aligned}$$

Similarly, noticing that $y_2^2y_3 = y_1y_3 \cdot y_3 = y_1y_3^2$ and $y_2^3 = y_2 \cdot y_1y_3 = y_1y_2 \cdot y_3 = y_0y_3 \cdot y_3 = y_0y_3^2$, we can use the second expression that defines ψ to compute

$$\begin{aligned} (\varphi \circ \psi)(q) &= \varphi([y_2 : y_3]) \\ &= [y_2^3 : y_2^2y_3 : y_2y_3^2 : y_3^3] \\ &= [y_0y_3^2 : y_1y_3^2 : y_2y_3^2 : y_3^3] \\ &= [y_0 : y_1 : y_2 : y_3]. \end{aligned}$$

The above calculation shows that φ and ψ are mutually inverse to each other, hence they are birational. Since they are both morphisms, they are isomorphisms. We conclude that Y is isomorphic to $\mathbb{P}^1$.

Solution 5.4. *A famous example: blow-up at a point.*

(1) All components of φ are homogeneous of degree 2. Given a point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$, if $x_0 \neq 0$ or $x_1 \neq 0$, then $x_0^2 \neq 0$ or $x_1^2 \neq 0$, hence at least one component of $\varphi(p)$ is non-zero, which implies $\varphi(p)$ is defined. When $\varphi(p)$ is defined, we need to check it is a point in Y. This can be verified by

$$\begin{aligned} y_0y_2 - y_1^2 &= (x_0^2)(x_1^2) - (x_0x_1)^2 = 0; \\ y_0y_4 - y_1y_3 &= (x_0^2)(x_1x_2) - (x_0x_1)(x_0x_2) = 0; \\ y_1y_4 - y_2y_3 &= (x_0x_1)(x_1x_2) - (x_1^2)(x_0x_2) = 0. \end{aligned}$$

This proves φ is a rational map.

(2) We first write down the formula for ψ, then prove ψ is a morphism, finally show that the two compositions of φ and ψ are identities.

The morphism $\psi : Y \longrightarrow \mathbb{P}^2$ is defined as follows: for every point $q = [y_0 : y_1 : y_2 : y_3 : y_4]$, let $\psi(q) = [y_0 : y_1 : y_3]$ or $[y_1 : y_2 : y_4]$. It is clear that both expressions in the definition of ψ are given by homogeneous polynomials of degree 1. When y_0, y_1 and y_3 are not simultaneously zero (e.g. $[1 : 0 : 0 : 0 : 0]$), then the first expression applies. When y_1, y_2 and y_4 are not simultaneously zero (e.g. $[0 : 0 : 0 : 0 : 1]$), then the second expression applies. Hence both expressions are defined on non-empty subsets of Y. For every point $q \in Y$, at least one of its coordinates is non-zero, which means at least one of two expressions is well-defined at q. And the image of q is clearly a point in $\mathbb{P}^2$, no matter which expression we use to compute the image.

We still need to show that the two expressions define the same image of q when they both apply. There are a few cases to consider. *Case 1:* if y_0, y_1 and y_3 are all non-zero, then set $\lambda = \frac{y_1}{y_0} = \frac{y_2}{y_1} = \frac{y_4}{y_3}$. Indeed, the three fractions are equal because of the defining equations of Y. Then $[y_0 : y_1 : y_3] = [\lambda y_0 : \lambda y_1 : \lambda y_3] = [y_1 : y_2 : y_4]$. *Case 2:* if $y_0 = 0$, then $y_1^2 = y_0y_2 = 0$ implies $y_1 = 0$. Since we assumed the expression $[y_0 : y_1 : y_3]$ is well-defined at q, we must have $y_3 \neq 0$. Then $y_2y_3 = y_1y_4 = 0$ implies $y_2 = 0$. Since we assumed the expression $[y_1 : y_2 : y_4]$ is well-defined at q, we must have $y_4 \neq 0$. Now $[y_0 : y_1 : y_3] = [0 : 0 : y_3] = [0 : 0 : y_4] = [y_1 : y_2 : y_4]$. *Case 3:* if $y_0 \neq 0$ and $y_1 = 0$, then $y_0y_2 = y_1^2 = 0$ implies $y_2 = 0$, and $y_0y_4 = y_1y_3 = 0$ implies $y_4 = 0$, then the expression $[y_1 : y_2 : y_4]$ is not defined at q. Hence this case cannot happen. *Case 4:* if $y_0 \neq 0$ and $y_1 \neq 0$ and $y_3 = 0$, then $y_0y_4 = y_1y_3 = 0$ implies $y_4 = 0$. Set $\lambda = \frac{y_1}{y_0} = \frac{y_2}{y_1}$. Then $[y_0 : y_1 : y_3] = [y_0 : y_1 : 0] = [\lambda y_0 : \lambda y_1 : 0] = [y_1 : y_2 : 0] = [y_1 : y_2 : y_4]$. In summary, we always have $[y_0 : y_1 : y_3] = [y_1 : y_2 : y_4]$. This finishes the proof of the fact that ψ is a morphism.

We compute the two compositions of φ and ψ. Given any point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ at which $\psi \circ \varphi$ is defined, we have

$$\begin{aligned}(\psi \circ \varphi)(p) &= \psi([x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2]) \\ &= \begin{cases} [x_0^2 : x_0x_1 : x_0x_2] = [x_0 : x_1 : x_2]; \\ [x_0x_1 : x_1^2 : x_1x_2] = [x_0 : x_1 : x_2]. \end{cases}\end{aligned}$$

Now pick any point $q = [y_0 : y_1 : y_2 : y_3 : y_4] \in Y$ at which $\varphi \circ \psi$ is defined. If we use the first expression to compute $\psi(q)$, then we have

$$\begin{aligned}(\varphi \circ \psi)(q) &= \varphi([y_0 : y_1 : y_3]) = [y_0^2 : y_0y_1 : y_1^2 : y_0y_3 : y_1y_3] \\ &= [y_0^2 : y_0y_1 : y_0y_2 : y_0y_3 : y_0y_4] = [y_0 : y_1 : y_2 : y_3 : y_4].\end{aligned}$$

If we use the second expression to compute $\psi(q)$, then we have

$$\begin{aligned}(\varphi \circ \psi)(q) &= \varphi([y_1 : y_2 : y_4]) = [y_1^2 : y_1y_2 : y_2^2 : y_1y_4 : y_2y_4] \\ &= [y_0y_2 : y_1y_2 : y_2^2 : y_2y_3 : y_2y_4] = [y_0 : y_1 : y_2 : y_3 : y_4].\end{aligned}$$

The above calculation shows that φ and ψ are mutually inverse rational maps. Hence Y and $\mathbb{P}^2$ are birational to each other. It follows that Y is rational.

(3) We have proved that ψ is a morphism. We first find all points $q \in Y$ such that $\psi(q) = [0 : 0 : 1]$. Let $q = [y_0 : y_1 : y_2 : y_3 : y_4] \in Y$. Then depending on which expression we use to compute $\psi(q)$, there are two possibilities. If $[y_0 : y_1 : y_3] = [0 : 0 : 1]$, then $y_0 = y_1 = 0$ and $y_3 \neq 0$. From $y_2y_3 = y_1y_4 = 0$ we obtain $y_2 = 0$. Hence $q = [0 : 0 : 0 : y_3 : y_4]$ for any $y_3 \neq 0$ and $y_4 \in \Bbbk$. Similarly, if $[y_1 : y_2 : y_4] = [0 : 0 : 1]$, then$y_1 = y_2 = 0$ and $y_4 \neq 0$. From $y_0y_4 = y_1y_3 = 0$ we obtain $y_0 = 0$. Hence $q = [0 : 0 : 0 : y_3 : y_4]$ for any $y_3 \in \Bbbk$ and $y_4 \neq 0$. Combining the two cases, all points $q \in Y$ satisfying $\psi(q) = [0 : 0 : 1]$ are given by points of the form $q = [0 : 0 : 0 : y_3 : y_4]$ where y_3 and y_4 not simultaneously zero.

Finally we need to show that ψ is surjective. We have seen that $[0 : 0 : 1]$ is in the image of ψ. For any point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ such that $p \neq [0 : 0 : 1]$, we claim that $p = \psi(q)$ for $q = [x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2]$. Indeed, when $p \neq [0 : 0 : 1]$, we have either $x_0 \neq 0$ or $x_1 \neq 0$. In such a case, we have checked in part (1) that $q = [x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2]$ is a well-defined point in Y. It remains to show $\psi(q) = p$. If $x_0 \neq 0$, then we can use the first expression of ψ to get $\psi(q) = [x_0^2 : x_0x_1 : x_0x_2] = [x_0 : x_1 : x_2] = p$. If $x_1 \neq 0$, then we can use the second expression of ψ to get $\psi(q) = [x_0x_1 : x_1^2 : x_1x_2] = [x_0 : x_1 : x_2] = p$. In summary, p is always in the image of ψ. Hence ψ is surjective.