Solution 5.1. Example: linear embedding and linear projection.

(1) All components are given by homogeneous polynomials of degree 1. For every point 
\([z_0 : z_1] \in \mathbb{P}^1\), we have either \(z_0 \neq 0\) or \(z_1 \neq 0\), hence \(\varphi([z_0 : z_1]) = [z_0 : z_1 : 0 : 0]\) has at least one non-zero coordinate, hence is clearly a point in \(\mathbb{P}^3\). Therefore \(\varphi\) is a morphism. It is not dominant, because for the projective algebraic set 
\(W = \mathbb{V}(z_2, z_3) \subseteq \mathbb{P}^3\), we have \(\varphi([z_0 : z_1]) \in W\) for every point \([z_0 : z_1] \in \mathbb{P}^1\).

(2) All components are given by homogeneous polynomials of degree 1. The map is 
not defined at every point in \(\mathbb{P}^3\), but for every point \([z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3\) with 
\(z_2 \neq 0\) or \(z_3 \neq 0\), its image \(\psi([z_0 : z_1 : z_2 : z_3]) = [z_2 : z_3]\) has at least one non-zero coordinate, and is clearly a point in \(\mathbb{P}^1\). Therefore \(\psi\) is a rational map. To see it is 
dominant, we first claim that \(\psi\) is surjective. In fact, for every point \([z_2 : z_3] \in \mathbb{P}^1\), 
we have that \([z_2 : z_3] = \psi([z_0 : z_1 : z_2 : z_3])\) for any choice of \(z_0, z_1 \in \mathbb{k}\). Since \(\psi\) 
is surjective, we can apply Lemma 5.16 and choose \(Z = \emptyset\) to conclude that \(\psi\) is 
dominant.

(3) The composition is not well-defined because for every \([z_0 : z_1] \in \mathbb{P}^1\), we have 
\((\psi \circ \varphi)([z_0 : z_1]) = \psi(\varphi([z_0 : z_1 : 0 : 0])) = [0 : 0]\) which is not a point in \(\mathbb{P}^1\). This 
shows that \(\psi \circ \varphi\) is nowhere well-defined, which violates the second condition in 
the definition of a rational map.

Solution 5.2. Example: the cooling tower.

(1) Assume we can write 
\(y_0y_3 - y_1y_2 = fg\) for some \(f, g \in \mathbb{k}[y_0, y_1, y_2, y_3]\). Since the 
polynomial \(y_0y_3 - y_1y_2\) has degree 1 in \(y_0\), the degrees of \(f\) and \(g\) in \(y_0\) should be 0 
and 1 respectively. Without loss of generality we assume \(f = y_0f_1 + f_0\) and \(g = g_0\), 
where \(f_1, f_0, g_0 \in \mathbb{k}[y_1, y_2, y_3]\). By comparing the coefficients of terms of degree 1 
and 0 in \(y_0\), we get \(f_1g_0 = y_3\) and \(f_0g_0 = -y_1y_2\). Therefore \(g_0\) is a common factor 
of \(y_3\) and \(-y_1y_2\), which has to be a constant. This implies \(g\) is a constant, hence 
\(y_0y_3 - y_1y_2\) is irreducible. Since it is a homogeneous polynomial, 
\(\mathbb{V}(y_0y_3 - y_1y_2)\) is a projective algebraic set. By Lemma 5.4, the principal ideal 
\(I = (y_0y_3 - y_1y_2)\) in \(\mathbb{k}[y_0, y_1, y_2, y_3]\) is a prime ideal. Hence by Lemma 4.17, 
\(\mathbb{V}(y_0y_3 - y_1y_2) = \mathbb{V}(I)\), which is a projective variety by Proposition 5.2.

(2) It is clear that all components of \(\varphi\) are given by homogeneous polynomials of 
degree 2. For any point \(p = [x_0 : x_1 : x_2] \in \mathbb{P}^2\), if \(x_0\) is non-zero, or \(x_1\) and \(x_2\) 
are simultaneously non-zero, the image \(\varphi(p)\) has at least one non-zero component. Hence \(\varphi\) 
is defined on a non-empty subset of \(\mathbb{P}^2\). To show its image is always in \(Y\), we find that 
\(y_0y_3 - y_1y_2 = (x_0^2)(x_1x_2) - (x_0x_1)(x_0x_2) = 0\). Therefore \(\varphi\) is a 
rational map.
To show that $\varphi$ is dominant, we observe that every point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ with $y_0 \neq 0$ is the image of the point $p = [y_0 : y_1 : y_2]$. Indeed, $\varphi(p) = [y_0^2 : y_0y_1 : y_0y_2 : y_1y_2] = [y_0^2 : y_0y_1 : y_0y_2 : y_0y_3] = [y_0 : y_1 : y_2 : y_3] = q$. Set $Z = V(y_0y_3 - y_1y_2, y_0)$, then $Z \subseteq Y$, and is strictly smaller than $Y$ (e.g. $[1 : 0 : 0 : 0] \in Y \setminus Z$). And every point $q \in Y \setminus Z$ is in the image of $\varphi$. By Lemma 5.16, $\varphi$ is dominant.

(3) We first realise that every component of $\psi$ is a homogeneous polynomial of degree 1. $\psi$ is well-defined at every point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ such that $y_0, y_1, y_2$ are not simultaneously zero (e.g. $[1 : 0 : 0 : 0]$ is such a point). Hence it is defined on a non-empty subset of $Y$. The image $\psi(q)$ is always a point in $\mathbb{P}^2$ if it is defined. Therefore $\psi$ is a rational map.

To show $\psi$ is dominant, we first observe that each point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ with $x_0 \neq 0$ is the image of the point $q = [x_0 : x_1 : x_2 : \frac{x_1x_2}{x_0}]$. Indeed, $q$ is a well-defined point since $x_0 \neq 0$, and $q \in Y$ since it satisfies the defining equation of $Y$. The expression that defines $\psi$ gives $\psi(q) = p$. If we set $Z = V(x_0)$, then $Z \subset \mathbb{P}^2$. Since every point in $\mathbb{P}^2 \setminus Z$ is in the image of $\psi$, we conclude that $\psi$ is dominant by Lemma 5.16.

(4) We show that $\varphi$ and $\psi$ are mutually inverse rational maps. For every point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ at which $\psi \circ \varphi$ is defined, we have $(\psi \circ \varphi)(p) = \psi([x_0^3 : x_0x_1 \cdot x_0x_2 : x_1x_2]) = [x_0^2 : x_0x_1 : x_0x_2] = [x_0 : x_1 : x_2] = p$. For every point $q = [y_0 : y_1 : y_2 : y_3] \in Y$ at which $\varphi \circ \psi$ is defined, we have $(\varphi \circ \psi)(q) = \varphi([y_0 : y_1 : y_2]) = [y_0^2 : y_0y_1 : y_0y_2 : y_1y_2] = [y_0^2 : y_0y_1 : y_0y_2 : y_0y_3] = [y_0 : y_1 : y_2 : y_3] = q$. Therefore $\varphi$ and $\psi$ are mutually inverse birational maps. It follows that $Y$ is birational to $\mathbb{P}^2$, hence $Y$ is rational.

Solution 5.3. Example: the projective twisted cubic.

(1) All components of $\varphi$ are homogeneous of the same degree 3. For every point $[u : v] \in \mathbb{P}^1$, we have either $u \neq 0$ or $v \neq 0$, therefore either $u^3 \neq 0$ or $v^3 \neq 0$, hence $\varphi([u : v]) = [u^3 : u^2v : uv^2 : v^3]$ is always a well-defined point. To show that $\varphi([u : v]) \in Y$, we need to check all defining polynomial of $Y$ are satisfied. Indeed, we have

$$y_0y_2 - y_1^2 = (u^3)(uv^2) - (u^2v)^2 = 0;$$
$$y_1y_3 - y_2^2 = (u^2v)(v^3) - (uv^2)^2 = 0;$$
$$y_0y_3 - y_1y_2 = (u^3)(v^3) - (u^2v)(uv^2) = 0.$$

We conclude that $\varphi$ is a morphism.

(2) We define $\psi : Y \to \mathbb{P}^1$ in the following way: for every point $[y_0 : y_1 : y_2 : y_3] \in Y$, let $\psi([y_0 : y_1 : y_2 : y_3]) = [y_0 : y_1]$ or $[y_2 : y_3]$. We first check that $\psi$ is a morphism.
Both expressions used to define \( \psi \) are given by homogeneous polynomials of degree 1. For any point \([y_0 : y_1 : y_2 : y_3]\), if either \(y_0\) or \(y_1\) is non-zero (e.g. \([1 : 0 : 0 : 0]\)), then the first expression applies; if either \(y_2\) or \(y_3\) is non-zero (e.g. \([0 : 0 : 0 : 1]\)), then the second expression applies. This shows that both expressions are defined on non-empty subsets of \(Y\). Moreover, for any point \([y_0 : y_1 : y_2 : y_3]\), at least one of its coordinates is non-zero, hence at least one of the expressions can be used to compute its image under \(\psi\), hence \(\psi\) is defined at every point in \(Y\). The image \(\psi(q)\) for any point \(q \in Y\) is clearly a point in \(\mathbb{P}^1\).

To show \(\psi\) is a morphism, it remains to show that, if the two expressions are both defined at a certain point \(q = [y_0 : y_1 : y_2 : y_3] \in Y\), then they give the same image. For such a point \(q\), we claim \(y_0 \neq 0\); otherwise \(y_0^2 = y_0 y_2 = 0\), which implies the first expression is invalid. Similarly, we claim \(y_3 \neq 0\); otherwise \(y_2^2 = y_1 y_3 = 0\), which implies the second expression is invalid. Therefore \(y_1 y_2 = y_0 y_3 \neq 0\), which implies \(y_1 \neq 0\) and \(y_2 \neq 0\). So all coordinates of \(q\) are non-zero. For such a point \(q\), we have \([y_0 : y_1] = [y_0 y_3 : y_1 y_3] = [y_1 y_2 : y_1 y_3] = [y_2 : y_3]\), hence both expressions give the same image of \(q\).

Finally we check that \(\varphi\) and \(\psi\) are mutually inverse to each other. Given any point \(p = [u : v] \in \mathbb{P}^1\), we have

\[
(\psi \circ \varphi)(p) = \psi([u^3 : u^2 v : uv^2 : v^3]) = \begin{cases} [u^3 : u^2 v] = [u : v]; \\ [uv^2 : v^3] = [u : v]. \end{cases}
\]

For any point \(q = [y_0 : y_1 : y_2 : y_3] \in Y\), we notice that \(y_0 y_1^2 = y_0 \cdot y_0 y_2 = y_0^2 y_2\) and \(y_1^2 = y_1 \cdot y_0 y_2 = y_0 \cdot y_1 y_2 = y_0 \cdot y_0 y_3 = y_0^2 y_3\). Therefore if we use the first expression that defines \(\psi\), we have

\[
(\varphi \circ \psi)(q) = \varphi([y_0 : y_1])
= [y_0^3 : y_0 y_1 : y_0 y_1^2 : y_1^3]
= [y_0^2 y_0 y_1 : y_0^2 y_2 : y_0^2 y_3]
= [y_0 : y_1 : y_2 : y_3].
\]

Similarly, noticing that \(y_2^2 y_3 = y_1 y_3 \cdot y_3 = y_1 y_3^2\) and \(y_3^2 = y_2 \cdot y_1 y_3 = y_1 y_2 \cdot y_3 = y_0 y_3 \cdot y_3 = y_0 y_3^2\), we can use the second expression that defines \(\psi\) to compute

\[
(\varphi \circ \psi)(q) = \varphi([y_2 : y_3])
= [y_2^3 : y_2^2 y_3 : y_2 y_3^2 : y_3^3]
= [y_0 y_3^2 : y_1 y_3^2 : y_2 y_3^2 : y_3^3]
= [y_0 : y_1 : y_2 : y_3].
\]

The above calculation shows that \(\varphi\) and \(\psi\) are mutually inverse to each other, hence they are birational. Since they are both morphisms, they are isomorphisms. We conclude that \(Y\) is isomorphic to \(\mathbb{P}^1\).
Solution 5.4. A famous example: blow-up at a point.

(1) All components of \( \varphi \) are homogeneous of degree 2. Given a point \( p = [x_0 : x_1 : x_2] \in \mathbb{P}^2 \), if \( x_0 \neq 0 \) or \( x_1 \neq 0 \), then \( x_0^2 \neq 0 \) or \( x_1^2 \neq 0 \), hence at least one component of \( \varphi(p) \) is non-zero, which implies \( \varphi(p) \) is defined. When \( \varphi(p) \) is defined, we need to check it is a point in \( Y \). This can be verified by

\[
\begin{align*}
y_0y_2 - y_1^2 &= (x_0^2)(x_1^2) - (x_0x_1)^2 = 0; \\
y_0y_4 - y_1y_3 &= (x_0^2)(x_1x_2) - (x_0x_1)(x_0x_2) = 0; \\
y_1y_4 - y_2y_3 &= (x_0x_1)(x_1x_2) - (x_1^2)(x_0x_2) = 0.
\end{align*}
\]

This proves \( \varphi \) is a rational map.

(2) We first write down the formula for \( \psi \), then prove \( \psi \) is a morphism, finally show that the two compositions of \( \varphi \) and \( \psi \) are identities.

The morphism \( \psi : Y \to \mathbb{P}^2 \) is defined as follows: for every point \( q = [y_0 : y_1 : y_2 : y_3 : y_4] \), let \( \psi(q) = [y_0 : y_1 : y_3] \) or \( [y_4 : y_2 : y_3] \). It is clear that both expressions in the definition of \( \psi \) are given by homogeneous polynomials of degree 1. When \( y_0, y_1 \) and \( y_3 \) are not simultaneously zero (e.g. \( [1 : 0 : 0 : 0] \)), then the first expression applies. When \( y_1, y_2 \) and \( y_4 \) are not simultaneously zero (e.g. \( [0 : 0 : 0 : 1] \)), then the second expression applies. Hence both expressions are defined on non-empty subsets of \( Y \). For every point \( q \in Y \), at least one of its coordinates is non-zero, which means at least one of two expressions is well-defined at \( q \). And the image of \( q \) is clearly a point in \( \mathbb{P}^2 \), no matter which expression we use to compute the image.

We still need to show that the two expressions define the same image of \( q \) when they both apply. There are a few cases to consider. Case 1: if \( y_0, y_1 \) and \( y_3 \) are all non-zero, then set \( \lambda = \frac{y_0}{y_1} = \frac{y_2}{y_1} = \frac{y_4}{y_1} \). Indeed, the three fractions are equal because of the defining equations of \( Y \). Then \( [y_0 : y_1 : y_3] = [\lambda y_0 : \lambda y_1 : \lambda y_3] = [y_1 : y_2 : y_4] \).

Case 2: if \( y_0 = 0 \), then \( y_1^2 = y_0y_2 = 0 \) implies \( y_1 = 0 \). Since we assumed the expression \( [y_0 : y_1 : y_3] \) is well-defined at \( q \), we must have \( y_3 \neq 0 \). Then \( y_2y_3 = y_1y_4 = 0 \) implies \( y_2 = 0 \). Since we assumed the expression \( [y_1 : y_2 : y_4] \) is well-defined at \( q \), we must have \( y_4 \neq 0 \). Now \( [y_0 : y_1 : y_3] = [0 : 0 : y_3] = [0 : 0 : y_4] = [y_1 : y_2 : y_4] \).

Case 3: if \( y_0 \neq 0 \) and \( y_1 = 0 \), then \( y_0y_2 = y_1^2 = 0 \) implies \( y_2 = 0 \), and \( y_0y_4 = y_1y_3 = 0 \) implies \( y_4 = 0 \), then the expression \( [y_1 : y_2 : y_4] \) is not defined at \( q \). Hence this case cannot happen.

Case 4: if \( y_0 \neq 0 \) and \( y_1 \neq 0 \) and \( y_3 = 0 \), then \( y_0y_4 = y_1y_3 = 0 \) implies \( y_4 = 0 \). Set \( \lambda = \frac{y_0}{y_1} = \frac{y_2}{y_1} \). Then \( [y_0 : y_1 : y_3] = [y_1 : y_1 : 0] = [\lambda y_1 : \lambda y_1 : 0] = [y_1 : y_2 : 0] = [y_1 : y_2 : y_4] \). In summary, we always have \( [y_0 : y_1 : y_3] = [y_1 : y_2 : y_4] \). This finishes the proof of the fact that \( \psi \) is a morphism.

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We compute the two compositions of $\varphi$ and $\psi$. Given any point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ at which $\psi \circ \varphi$ is defined, we have
\[
(\psi \circ \varphi)(p) = \psi([x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2])
\]
\[
= \begin{cases} 
[x_0^2 : x_0x_1 : x_0x_2] = [x_0 : x_1 : x_2]; \\
[x_0x_1 : x_1^2 : x_1x_2] = [x_0 : x_1 : x_2].
\end{cases}
\]
Now pick any point $q = [y_0 : y_1 : y_2 : y_3 : y_4] \in Y$ at which $\varphi \circ \psi$ is defined. If we use the first expression to compute $\psi(q)$, then we have
\[
(\varphi \circ \psi)(q) = \varphi([y_0 : y_1 : y_3]) = [y_0^2 : y_0y_1 : y_1^2 : y_0y_3 : y_1y_3]
\]
\[
= [y_0^2 : y_0y_1 : y_0y_2 : y_0y_3 : y_0y_4] = [y_0 : y_1 : y_2 : y_3 : y_4].
\]
If we use the second expression to compute $\psi(q)$, then we have
\[
(\varphi \circ \psi)(q) = \varphi([y_1 : y_2 : y_4]) = [y_1^2 : y_1y_2 : y_2^2 : y_1y_4 : y_2y_4]
\]
\[
= [y_0y_2 : y_1y_2 : y_2^2 : y_2y_3 : y_2y_4] = [y_0 : y_1 : y_2 : y_3 : y_4].
\]
The above calculation shows that $\varphi$ and $\psi$ are mutually inverse rational maps. Hence $Y$ and $\mathbb{P}^2$ are birational to each other. It follows that $Y$ is rational.

(3) We have proved that $\psi$ is a morphism. We first find all points $q \in Y$ such that $\psi(q) = [0 : 0 : 1]$. Let $q = [y_0 : y_1 : y_2 : y_3 : y_4] \in Y$. Then depending on which expression we use to compute $\psi(q)$, there are two possibilities. If $[y_0 : y_1 : y_3] = [0 : 0 : 1]$, then $y_0 = y_1 = 0$ and $y_3 \neq 0$. From $y_2y_3 = y_1y_4 = 0$ we obtain $y_2 = 0$. Hence $q = [0 : 0 : 0 : y_3 : y_4]$ for any $y_3 \neq 0$ and $y_4 \in \mathbb{K}$. Similarly, if $[y_1 : y_2 : y_4] = [0 : 0 : 1]$, then $y_1 = y_2 = 0$ and $y_4 \neq 0$. From $y_0y_4 = y_1y_3 = 0$ we obtain $y_0 = 0$. Hence $q = [0 : 0 : 0 : y_3 : y_4]$ for any $y_3 \in \mathbb{K}$ and $y_4 \neq 0$. Combining the two cases, all points $q \in Y$ satisfying $\psi(q) = [0 : 0 : 1]$ are given by points of the form $q = [0 : 0 : 0 : y_3 : y_4]$ where $y_3$ and $y_4$ not simultaneously zero.

Finally we need to show that $\psi$ is surjective. We have seen that $[0 : 0 : 1]$ is in the image of $\psi$. For any point $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ such that $p \neq [0 : 0 : 1]$, we claim that $p = \psi(q)$ for $q = [x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2]$. Indeed, when $p \neq [0 : 0 : 1]$, we have either $x_0 \neq 0$ or $x_1 \neq 0$. In such a case, we have checked in part (1) that $q = [x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2]$ is a well-defined point in $Y$. It remains to show $\psi(q) = p$. If $x_0 \neq 0$, then we can use the first expression of $\psi$ to get $\psi(q) = [x_0^2 : x_0x_1 : x_0x_2] = [x_0 : x_1 : x_2] = p$. If $x_1 \neq 0$, then we can use the second expression of $\psi$ to get $\psi(q) = [x_0x_1 : x_1^2 : x_1x_2] = [x_0 : x_1 : x_2] = p$. In summary, $p$ is always in the image of $\psi$. Hence $\psi$ is surjective.