Solutions to Exercise Sheet 6

Solution 6.1. *Example: the cooling tower, revisited.*

(1) We can get the standard affine pieces $Y_i = Y \cap U_i$ by setting $y_i = 1$. Therefore the standard affine pieces of Y are given by $Y_0 = \mathbb{V}_a(y_3 - y_1y_2)$, $Y_1 = \mathbb{V}_a(y_0y_3 - y_2)$, $Y_2 = \mathbb{V}_a(y_0y_3 - y_2)$ and $Y_3 = \mathbb{V}_a(y_0 - y_1y_2)$.

(2) We proved in Exercise 5.2 that Y is birational to $\mathbb{P}^2$. By Proposition 6.21 and Example 6.18, we have $\Bbbk(X) \cong \Bbbk(\mathbb{P}^2) \cong \Bbbk(x_1, x_2)$.

Solution 6.2. *Example: irreducible cubic curves.*

(1) We claim that $y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$ is an irreducible polynomial. Use contradiction. Assume $y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = f(x, y)g(x, y)$ for non-constant polynomials $f, g \in \Bbbk[x, y]$. Since the left-hand side has degree 2 in y, the degrees of f and g in y must be either 2 and 0, or 1 and 1. In the first case we can write

$$y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = (y^2 f_2(x) + y f_1(x) + f_0(x)) \cdot g(x).$$

Comparing coefficients of y^2 we find $f_2(x)g(x) = 1$, hence $g(x)$ must be a constant. Contradiction. In the second case we can write

$$y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = (y f_1(x) + f_0(x)) \cdot (y g_1(x) + g_0(x)).$$

Comparing coefficients of y^2 we find $f_1(x)g_1(x) = 1$. Without loss of generality we can assume $f_1(x) = g_1(x) = 1$. Comparing coefficients of y we find $f_0(x) + g_0(x) = 0$. Comparing constant terms we find $-(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = f_0(x)g_0(x) = -f_0(x)^2$, hence $f_0(x)^2 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, which is also a contradiction since the right-hand side is not a square. So we conclude that $y^2-(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is irreducible. By Lemma 5.4 we know $I = (y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3))$ is a prime ideal. By Proposition 2.15 we know X is an irreducible algebraic set, i.e. an affine variety.

(2) Using z as the extra variable, the projective closure is given by $\overline{X} = \mathbb{V}_p(y^2z - (x - \lambda_1z)(x - \lambda_2z)(x - \lambda_3z))$. To find points at infinity, we set $z = 0$ to get $-x^3 = 0$. It follows that $x = 0$, hence the only point at infinity for X is given by $[x : y : z] = [0 : 1 : 0]$. One direction of Proposition 6.12 shows that the projective closure of a non-empty affine variety is a projective variety. Hence by part (1), we conclude that $\overline{X}$ is a projective variety.

Solution 6.3. *A caution for the projective closure.*

(1) The homogenisation of f_1 and f_2 are given by $\overline{f_1} = wy - x^2$ and $\overline{f_2} = w^2z - x^3$.

(2) We first claim $y^2 - xz \in I = (y - x^2, z - x^3)$. This can be seen by realising $y^2 - xz = (y^2 - x^4) + (x^4 - xz) = (y - x^2)(y + x^2) - x(z - x^3)$ which is a sum of a term with $y - x^2$ as a factor and a term with $z - x^3$ as a factor. Since $y^2 - xz$ is an element in I, by Definition 6.5, the homogenisation of $y^2 - xz$ is an element in $\overline{I}$. However, since $y^2 - xz$ is already homogeneous, its homogenisation is still $y^2 - xz$. Therefore $y^2 - xz \in \overline{I}$.

We prove that $y^2 - xz \notin (\overline{f_1}, \overline{f_2})$. Use contradiction. Assume we can write $y^2 - xz = \overline{f_1} \cdot g_1 + \overline{f_2} \cdot g_2 = (wy - x^2) \cdot g_1 + (w^2 z - x^3) \cdot g_2$ for some $g_1, g_2 \in \Bbbk[w, x, y, z]$. There are many different ways to find a contradiction. Here is one approach: when $w = x = 0$ and $y = z = 1$, the left-hand side is 1 while the right-hand side is 0, which is a contradiction.

Finally we prove that $\overline{X} \neq \mathbb{V}_p(\overline{f_1}, \overline{f_2})$. There are also many different approaches to this. Here is one of them: On one hand, we can verify directly that $\overline{f_1} = 0$ and $\overline{f_2} = 0$ at the point $[w : x : y : z] = [0 : 0 : 1 : 1]$, hence $[0 : 0 : 1 : 1] \in \mathbb{V}_p(\overline{f_1}, \overline{f_2})$. On the other hand, since $\overline{X} = \mathbb{V}_p(\overline{I})$, a point in $\overline{X}$ has to be a solution to every homogeneous polynomial in $\overline{I}$, in particular, it has to be a solution to the polynomial $y^2 - xz$ by what we just proved. We can check directly that the point $[w : x : y : z] = [0 : 0 : 1 : 1]$ is not a solution to this polynomial, hence $[0 : 0 : 1 : 1] \notin \overline{X}$. This finishes the proof.

Indeed, one can see that the value of z is irrelavant. For any $\lambda \in \Bbbk$, the point $[w : x : y : z] = [0 : 0 : 1 : \lambda]$ would do the trick.

Solution 6.4. *Geometric interpretation of the projective closure.*

(1) We need to show that $f(p) = 0$ for every point $p \in X$. Let $p = (a_1, \cdots, a_n) \in X$, where $a_1, \cdots, a_n \in \Bbbk$ are the non-homogeneous coordinates of p as a point in $\mathbb{A}^n \cong U_0$. Then as a point in $\mathbb{P}^n$, the homogeneous coordinates of p can be given by $p = [1 : a_1 : \cdots : a_n]$. Since $X \subseteq W$, we have $p \in W$, therefore $g(p) = 0$. In other words, $g(1, a_1, \cdots, a_n) = 0$. Therefore we have $f(a_1, \cdots, a_n) = g(1, a_1, \cdots, a_n) = 0$, which proves $f(p) = 0$. Since p is an arbitrary point in X, we conclude that $f \in \mathbb{I}_a(X)$.

(2) We assume g is a homogeneous polynomial with $\deg g = d$. Assume that z_0^k is the highest power dividing g, then k is a non-negative integer, and each term in g has a factor of z_0^k. We collect terms in g which have the degree with respect to z_0, so we can write

$$g = z_0^k \cdot f_{d-k} + z_0^{k+1} \cdot f_{d-k-1} + \cdots + z_0^{d-1} \cdot f_1 + z_0^d \cdot f_0$$

where $f_i \in \Bbbk[z_1, \cdots, z_n]$ is homogeneous of degree i for $i = 0, 1, \cdots, d - k$, and $f_{d-k} \neq 0$. Since f is the dehomogenisation of g with respect to z_0, we have

$$f = f_{d-k} + f_{d-k-1} + \cdots + f_1 + f_0$$

which is precisely the homogeneous decomposition of f. We observe that $\deg f = d-k$. Since $\overline{f}$ is the homogenisation of f with respect to z_0, we have

$$\overline{f} = f_{d-k} + z_0 \cdot f_{d-k-1} + \cdots + z_0^{d-k-1} \cdot f_1 + z_0^{d-k} \cdot f_0.$$

Comparing the formula for g and $\overline{f}$, we find out that $g = z_0^k \cdot \overline{f}$.

Now we prove $g \in \overline{I}$. Since $f \in \mathbb{I}_a(X)$ by part (1), we have $\overline{f} \in \overline{I}$ by Definition 6.5. Since $\overline{I}$ is an ideal, we have $g = z_0^k \cdot \overline{f} \in \overline{I}$.

Since g is an arbitrary homogeneous polynomial in $\mathbb{I}_p(W)$, we conclude that every homogeneous polynomial in the ideal $\mathbb{I}_p(W)$ is a homogeneous polynomial in the ideal $\overline{I}$. It follows that $\mathbb{V}_p(\mathbb{I}_p(W)) \supseteq \mathbb{V}_p(\overline{I})$. We have $\mathbb{V}_p(\mathbb{I}_p(W)) = W$ by Proposition 5.2, and $\mathbb{V}_p(\overline{I}) = \overline{X}$ by Definition 6.5. Therefore $W \supseteq \overline{X}$.

(3) We proved in parts (1) and (2) that every projective algebraic set W that contains X must contain $\overline{X}$. Since $\overline{X}$ itself is also a projective algebraic set that contains X (it is X together with points at infinity), we conclude that $\overline{X}$ is the smallest one having this property.