Solution 7.1. Examples of affine varieties.

(1) The singular points are defined by \( f = 0 \) and the two partial derivatives \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \). We have \( \frac{\partial f}{\partial x} = 6x(x^2 + y^2)^2 - 8xy^2 = 2x \cdot (3(x^2 + y^2)^2 - 4y^2) \) and \( \frac{\partial f}{\partial y} = 6y(x^2 + y^2)^2 \cdot 2y - 8x^2y = 2y \cdot (3(x^2 + y^2)^2 - 4x^2) \). If \( x = 0 \) or \( y = 0 \), then \( f = 0 \) forces \( x = y = 0 \). The point \((0,0)\) satisfies all equations hence is a singular point. If neither \( x \) nor \( y \) is 0, then we have \( 3(x^2 + y^2)^2 = 4x^2 = 4y^2 \), hence \( 3(x^2 + y^2)^2 = 4x^2 \) which implies \( x^2 = \frac{1}{3} = y^2 \). But then \( f = \left( \frac{1}{3} + \frac{1}{3} \right)^3 - 4 \cdot \frac{1}{3} \cdot \frac{1}{3} \neq 0 \). Therefore the only singular point is \((0,0)\).

(2) The singular points are defined by \( f = xy - z^2 = 0 \), and \( \frac{\partial f}{\partial x} = y^2 = 0 \), \( \frac{\partial f}{\partial y} = 2xy = 0 \), \( \frac{\partial f}{\partial z} = -2z = 0 \). From the second and fourth equations we have \( y = z = 0 \). No matter what value \( x \) takes, \((x,y,z) = (x,0,0)\) always satisfies all the four equations. Therefore the singular points of \( \mathbb{V}(f) \) are all points of the form \((x,0,0)\).

(3) The singular points are given by \( f = xy + x^3 + y^3 = 0 \), and \( \frac{\partial f}{\partial x} = y + 3x^2 = 0 \), \( \frac{\partial f}{\partial y} = x + 3y^2 = 0 \), \( \frac{\partial f}{\partial z} = 0 \). From \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \) we get \( x = -3y^2 = -27x^4 \), hence \( x = 0 \) or \( x^3 = -\frac{1}{27} \). If \( x = 0 \), then \( f = 0 \) forces \( y = 0 \). It is clear that every point of the form \((x,y,z) = (0,0,z)\) is a solution to all the required equations hence is a singular point on \( \mathbb{V}(f) \). If \( x \neq 0 \), then \( x^3 = -\frac{1}{27} \). Then we have \( f = xy + x^3 + y^3 = x(-3x^2) + x^3 + (-3x^2)^3 = -3x^3 + x^3 - 27x^6 = \frac{1}{9} - \frac{1}{27} - \frac{1}{27} = \frac{1}{27} \neq 0 \). Contradiction. Therefore \((x,y,z) = (0,0,z)\) are the only singular points of \( \mathbb{V}(f) \).

(4) At every point \( p = (x,y,z) \in X \), we consider the matrix \( M_p \) given by the partial derivatives

\[
M_p = \begin{pmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{pmatrix}.
\]

It is clear that the two rows of \( M_p \) are linearly independent, therefore \( \text{rank} \ M_p = 2 \) for every \( p \in X \). It follows that \( \dim T_pX = 3 - \text{rank} \ M_p = 1 \) for every \( p \in X \). Therefore \( \dim X = 1 \) and \( \dim T_pX = \dim X \) for every \( p \in X \). By Definition 7.13, \( X \) is non-singular at every point \( p \in X \).

Solution 7.2. Example of projective varieties.

(1) The standard affine piece \( X_0 = X \cap U_0 \) is given by setting \( x = 1 \) in \( f \). Hence \( X_0 = \mathbb{V}(f_0) \) where \( f_0 = y - z^2 \). For any point \((y,z) \in X_0 \), \( \frac{\partial f_0}{\partial y} = 1 \) which never vanishes. Therefore \( X_0 \) does not have any singular point, hence is non-singular.

(2) The set of points in \( X \setminus X_0 \) is given by \( \{ [x : y : z] \in X \mid x = 0 \} \). When \( x = 0 \), \( f = xy - z^2 = 0 \) implies \( z = 0 \). Hence the only point in \( X \setminus X_0 \) is \( p = [x : y : z] = [0 : 1 : 0] \). This point is in the standard affine piece \( X_1 = X \cap U_1 \) because its \( y \)-coordinate is non-zero. The standard affine piece \( X_1 \) is obtained by setting
\[ y = 1 \] hence \( X_1 = \mathbb{V}_a(f_1) \) where \( f_1 = x - z^2 \). The point \( p = [0 : 1 : 0] \) has non-homogeneous coordinates \( p = (0,0) \) in the standard affine piece \( X_1 \). To check whether \( X_1 \) is singular at \( p = (0,0) \), we need to compute the partial derivatives of the defining equation \( f_1 \). Notice that \( \frac{\partial f_1}{\partial x} = 1 \) which does not vanish at \( p \).

We conclude that \( p \) is a non-singular point of \( X_1 \), hence by Definition 7.8, \( p \) is a non-singular point of \( X \).

Parts (1) and (2) together show that \( X = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2 \) is non-singular.

(3) We first consider the standard affine piece \( X_0 = X \cap U_0 \). By setting \( x = 1 \), we get \( X_0 = \mathbb{V}(f_0) \subseteq \mathbb{A}^2 \) where \( f_0 = z + yz + y^3z + 1 + y^4 \). To find singular points in \( X_0 \), we need to consider the equations

\[
\begin{align*}
f_0 &= z + yz + y^3z + 1 + y^4 = 0; \\
\frac{\partial f_0}{\partial y} &= z + 3y^2z + 4y^3 = 0; \\
\frac{\partial f_0}{\partial z} &= 1 + y + y^3 = 0.
\end{align*}
\]

We now solve the system. From the first equation we observe that \( f_0 = z(1 + y + y^3) + (1 + y^4) = 0 \). Together with the third equation we find that \( 1 + y^4 = 0 \). I claim that the two equations \( 1 + y + y^3 = 0 \) and \( 1 + y^4 = 0 \) do not have a common solution for \( y \). There are many ways to prove the claim. One possible way is to use the Euclidean division. We divide \( y^4 + 1 \) by \( y^3 + y + 1 \) to get

\[ y^4 + 1 = y(y^3 + y + 1) - (y^2 + y - 1), \]

which implies \( y^2 + y - 1 = 0 \). We further divide \( y^3 + y + 1 \) by \( y^2 + y - 1 \) to get

\[ y^3 + y + 1 = (y - 1)(y^2 + y - 1) + 3y, \]

which implies \( 3y = 0 \) hence \( y = 0 \). Therefore if the two equations have a common solution for \( y \) then we must have \( y = 0 \), which is not a solution. This proves the claim, which implies that \( X_0 \) is non-singular.

Finally we need to check whether the points in \( X \setminus X_0 \) are singular points. To find all points in \( X \setminus X_0 \), we set \( x = 0 \) in \( f = 0 \). Then we get \( y^3z + y^4 = 0 \), which implies \( y = 0 \) or \( y + z = 0 \). Therefore there are two points in \( X \setminus X_0 \), given by \( p_1 = [0 : 0 : 1] \) and \( p_2 = [0 : -1 : 1] \) respectively. To check whether they are singular points, we need to find a standard affine piece which contain them. Since the \( z \)-coordinates of \( p_1 \) and \( p_2 \) are non-zero, we can choose \( X_2 = X \cap U_2 \). The standard affine piece \( X_2 = \mathbb{V}(f_2) \) where \( f_2 = x^3 + x^2y + y^3 + x^4 + y^4 \). The non-homogeneous coordinates of \( p_1 \) and \( p_2 \) are given by \( p_1 = (0,0) \) and \( p_2 = (0,-1) \).
respectively. The partial derivatives of \( f \) are

\[
\frac{\partial f_2}{\partial x} = 3x^2 + 2xy + 4x^3; \\
\frac{\partial f_2}{\partial y} = x^2 + 3y^2 + 4y^3.
\]

It is easy to see that at the point \( p_1 = (0, 0) \), we have \( f_2(p_1) = \frac{\partial f_2}{\partial x}(p_1) = \frac{\partial f_2}{\partial y}(p_1) = 0 \). Therefore \( p_1 \) is a singular point on \( X_2 \). At the point \( p_2 = (0, -1) \), we have \( \frac{\partial f_2}{\partial y}(p_2) = -1 \neq 0 \). Therefore \( p_2 \) is a non-singular point on \( X_2 \). By Definition 7.8, the only singular point of \( X \) is \( p_1 = [0 : 0 : 1] \).

**Solution 7.3. Example: plane cubics.** There are three cases to deal with in this question. Most of the calculations are the same in all the three cases. First of all we look at a standard affine piece of \( X = \mathbb{V}(f) \subseteq \mathbb{P}^2 \). You can choose any standard affine piece of \( X \) to start with. For example, we choose the standard affine piece \( X_2 = X \cap U_2 \), which is given by setting \( z = 1 \) in \( f \). Therefore we have

\[
X_2 = \mathbb{V}(y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)) \subseteq \mathbb{A}^2.
\]

To find the singular points on \( X_2 \), we need to solve the system

\[
y^2 - (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = 0; \\
-(x - \lambda_2)(x - \lambda_3) - (x - \lambda_1)(x - \lambda_3) - (x - \lambda_1)(x - \lambda_3) = 0; \\
2y = 0.
\]

The third equation implies \( y = 0 \), then the first equation implies \( x = \lambda_1 \) or \( \lambda_2 \) or \( \lambda_3 \). Now there is some difference in the three cases.

1. If \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are distinct, then it is clear that none of them is a solution to the second equation. Therefore \( X_2 \) is non-singular in this case.

2. If two of the three are equal, say, \( \lambda_1 = \lambda_2 \neq \lambda_3 \), then it is clear that \( x = \lambda_1 \) (or \( \lambda_2 \)) is a solution to the second equation while \( x = \lambda_3 \) is not a solution. Therefore \( X_2 \) has a singular point \( (\lambda_1, 0) \), which has homogeneous coordinates \( [\lambda_1 : 0 : 1] \) as a point in \( X \).

3. If all the three are equal, then \( x = \lambda_1 \) (or \( \lambda_2 \) or \( \lambda_3 \)) is a solution to the second equation. Therefore \( X_2 \) has a singular point \( (\lambda_1, 0) \), which has homogeneous coordinates \( [\lambda_1 : 0 : 1] \) as a point in \( X \).

It remains to consider the points in \( X \setminus X_2 \). To find these points we set \( z = 0 \) in the equation \( f = 0 \). We get \( -x^3 = 0 \) hence \( x = 0 \). Therefore the only point in \( X \setminus X_2 \) is \( p = [x : y : z] = [0 : 1 : 0] \). Since the \( y \)-coordinate of \( p \) is non-zero, it is a point in the standard affine piece \( X_1 = X \cap U_1 \), given by the non-homogeneous coordinates \( p = (0, 0) \).
To write down the defining polynomial for $X_1$ we set $y = 1$ and get $X_1 = \mathbb{V}(f_1) \subseteq \mathbb{A}^2$ where
\[ f_1 = z - (x - \lambda_1 z)(x - \lambda_2 z)(x - \lambda_3 z). \]
Its partial derivative with respect to $z$ is given by
\[ \frac{\partial f_1}{\partial z} = 1 + \lambda_1(x - \lambda_2 z)(x - \lambda_3 z) + \lambda_2(x - \lambda_1 z)(x - \lambda_3 z) + \lambda_3(x - \lambda_1 z)(x - \lambda_2 z). \]
It is clear that at the point $p = (0, 0)$, we have $\frac{\partial f_1}{\partial z}(p) = 1 \neq 0$. Therefore $p = (0, 0)$ is a non-singular point of $X_1$, hence $p = [0 : 1 : 0]$ is a non-singular point of $X$. This holds in all the three cases. We have the following conclusion:

1. If $\lambda_1$, $\lambda_2$ and $\lambda_3$ are distinct, $X$ is non-singular.
2. If two of the three are equal, say, $\lambda_1 = \lambda_2 \neq \lambda_3$, then $X$ has a unique singular point $[\lambda_1 : 0 : 1]$.
3. If all the three are equal, then $X$ has a unique singular point $[\lambda_1 : 0 : 1]$.

**Solution 7.4. Example: projective twisted cubic.** We first consider the standard affine piece $Y_0 = Y \cap U_0$. By setting $z_0 = 1$ we get
\[ Y_0 = \mathbb{V}_a(y_2 - y_1^2, y_1 y_3 - y_2^2, y_3 - y_1 y_2). \]
To find the dimension of the tangent space at any point $p = (y_1, y_2, y_3)$, we consider the matrix of partial derivatives:
\[ M_p = \begin{pmatrix} -2y_1 & 1 & 0 \\ y_3 & -2y_2 & y_1 \\ -y_2 & -y_1 & 1 \end{pmatrix}. \]
We need to find rank $M_p$. First we compute the determinant of $M_p$:
\[ \det M_p = 4y_1 y_2 - y_1 y_2 - y_3 - 2y_1^3 = 4y_1 y_2 - y_1 y_2 - 2y_1 y_2 = 0. \]
Therefore rank $M_p \leq 2$. Notice that the first and third rows of $M_p$ are linearly independent (or the second and third columns). Therefore rank $M_p = 2$, which implies $\dim T_p Y_0 = 1$ at every $p \in Y_0$. It follows that $Y_0$ is non-singular and $\dim Y = \dim Y_0 = 1$.

Now we consider the points in $Y \setminus Y_0$. Let $p = [y_0 : y_1 : y_2 : y_3]$ be such a point, then $y_0 = 0$, which implies $y_1^2 = y_0 y_2 = 0$ and $y_2^2 = y_1 y_3 = 0$. Therefore the only point $p \in Y \setminus Y_0$ is given by $p = [0 : 0 : 0 : 1]$. To determine whether $p$ is a singular point, we need to look at the standard affine piece $Y_3 = Y \cap U_3$. We could perform a similar calculation as above to show that $Y_3$ is non-singular. More precisely, we have
\[ Y_3 = \mathbb{V}_a(y_0 y_2 - y_1^2, y_1 y_2^2, y_0 - y_1 y_2). \]
For any point $q = (y_0, y_1, y_2) \in Y_3$, the matrix

$$M_q = \begin{pmatrix} y_2 & -2y_1 & y_0 \\ 0 & 1 & -2y_2 \\ 1 & -y_2 & -y_1 \end{pmatrix}.$$ 

We notice that

$$\det M_q = -y_1 y_2 + 4y_1 y_2 - y_0 - 2y_2^3 = -y_1 y_2 + 4y_1 y_2 - y_1 y_2 - 2y_1 y_2 = 0.$$ 

Therefore rank $M_q \leq 2$. Moreover the second and the third rows are linearly independent, hence rank $M_q = 2$ for every $q \in Y_3$. It follows that $Y_3$ is non-singular. To summarise, $Y$ is non-singular and has dimension 1.