Solutions to Exercise Sheet 7

Solution 7.1. *Examples of affine varieties.*

(1) The singular points are defined by $f = 0$ and the two partial derivatives $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. We have $\frac{\partial f}{\partial x} = 6x(x^2 + y^2)^2 - 8xy^2 = 2x \cdot (3(x^2 + y^2)^2 - 4y^2)$ and $\frac{\partial f}{\partial y} = 6y(x^2 + y^2)^2 \cdot 2y - 8x^2 y = 2y \cdot (3(x^2 + y^2)^2 - 4x^2)$. If $x = 0$ or $y = 0$, then $f = 0$ forces $x = y = 0$. The point $(0, 0)$ satisfies all equations hence is a singular point. If neither x nor y is 0, then we have $3(x^2+y^2)^2 = 4x^2 = 4y^2$, hence $3(x^2+x^2)^2 = 4x^2$ which implies $x^2 = \frac{1}{3} = y^2$. But then $f = (\frac{1}{3} + \frac{1}{3})^3 - 4 \cdot \frac{1}{3} \cdot \frac{1}{3} \neq 0$. Therefore the only singular point is $(0, 0)$.

(2) The singular points are defined by $f = xy^2 - z^2 = 0$, and $\frac{\partial f}{\partial x} = y^2 = 0$, $\frac{\partial f}{\partial y} = 2xy = 0$, $\frac{\partial f}{\partial z} = -2z = 0$. From the second and fourth equations we have $y = z = 0$. No matter what value x takes, $(x, y, z) = (x, 0, 0)$ always satisfies all the four equations. Therefore the singular points of $\mathbb{V}(f)$ are all points of the form $(x, 0, 0)$.

(3) The singular points are given by $f = xy + x^3 + y^3 = 0$, and $\frac{\partial f}{\partial x} = y + 3x^2 = 0$, $\frac{\partial f}{\partial y} = x + 3y^2 = 0$, $\frac{\partial f}{\partial z} = 0$. From $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ we get $x = -3y^2 = -27x^4$, hence $x = 0$ or $x^3 = -\frac{1}{27}$. If $x = 0$, then $f = 0$ forces $y = 0$. It is clear that every point of the form $(x, y, z) = (0, 0, z)$ is a solution to all the required equations hence is a singular point on $\mathbb{V}(f)$. If $x \neq 0$, then $x^3 = -\frac{1}{27}$. Then we have $f = xy+x^3+y^3 = x(-3x^2)+x^3+(-3x^2)^3 = -3x^3+x^3-27x^6 = \frac{1}{9}-\frac{1}{27}-\frac{1}{27} = \frac{1}{27} \neq 0$. Contradiction. Therefore $(x, y, z) = (0, 0, z)$ are the only singular points of $\mathbb{V}(f)$.

(4) At every point $p = (x, y, z) \in X$, we consider the matrix M_p given by the partial derivatives

$$M_p = \begin{pmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{pmatrix}.$$

It is clear that the two rows of M_p are linearly independent, therefore $\operatorname{rank} M_p = 2$ for every $p \in X$. It follows that $\dim T_p X = 3 - \operatorname{rank} M_p = 1$ for every $p \in X$. Therefore $\dim X = 1$ and $\dim T_p X = \dim X$ for every $p \in X$. By Definition 7.13, X is non-singular at every point $p \in X$.

Solution 7.2. *Example of projective varieties.*

(1) The standard affine piece $X_0 = X \cap U_0$ is given by setting $x = 1$ in f. Hence $X_0 = \mathbb{V}(f_0)$ where $f_0 = y - z^2$. For any point $(y, z) \in X_0$, $\frac{\partial f_0}{\partial y} = 1$ which never vanishes. Therefore X_0 does not have any singular point, hence is non-singular.

(2) The set of points in $X \backslash X_0$ is given by $\{[x : y : z] \in X \mid x = 0\}$. When $x = 0$, $f = xy - z^2 = 0$ implies $z = 0$. Hence the only point in $X \backslash X_0$ is $p = [x : y : z] = [0 : 1 : 0]$. This point is in the standard affine piece $X_1 = X \cap U_1$ because its y-coordinate is non-zero. The standard affine piece X_1 is obtained by setting

$y = 1$ hence $X_1 = \mathbb{V}_a(f_1)$ where $f_1 = x - z^2$. The point $p = [0 : 1 : 0]$ has non-homogeneous coordinates $p = (0, 0)$ in the standard affine piece X_1. To check whether X_1 is singular at $p = (0, 0)$, we need to compute the partial derivatives of the defining equation f_1. Notice that $\frac{\partial f_1}{\partial x} = 1$ which does not vanish at p. We conclude that p is a non-singular point of X_1, hence by Definition 7.8, p is a non-singular point of X.

Parts (1) and (2) together show that $X = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2$ is non-singular.

(3) We first consider the standard affine piece $X_0 = X \cap U_0$. By setting $x = 1$, we get $X_0 = \mathbb{V}(f_0) \subseteq \mathbb{A}^2$ where $f_0 = z + yz + y^3z + 1 + y^4$. To find singular points in X_0, we need to consider the equations

$$\begin{aligned} f_0 = z + yz + y^3z + 1 + y^4 = 0; \\ \frac{\partial f_0}{\partial y} = z + 3y^2z + 4y^3 = 0; \\ \frac{\partial f_0}{\partial z} = 1 + y + y^3 = 0. \end{aligned}$$

We now solve the system. From the first equation we observe that $f_0 = z(1 + y + y^3) + (1 + y^4) = 0$. Together with the third equation we find that $1 + y^4 = 0$. I claim that the two equations $1 + y + y^3 = 0$ and $1 + y^4 = 0$ do not have a common solution for y. There are many ways to prove the claim. One possible way is to use the Euclidean division. We divide $y^4 + 1$ by $y^3 + y + 1$ to get

$$y^4 + 1 = y(y^3 + y + 1) - (y^2 + y - 1),$$

which implies $y^2 + y - 1 = 0$. We further divide $y^3 + y + 1$ by $y^2 + y - 1$ to get

$$y^3 + y + 1 = (y - 1)(y^2 + y - 1) + 3y,$$

which implies $3y = 0$ hence $y = 0$. Therefore if the two equations have a common solution for y then we must have $y = 0$, which is not a solution. This proves the claim, which implies that X_0 is non-singular.

Finally we need to check whether the points in $X \backslash X_0$ are singular points. To find all points in $X \backslash X_0$, we set $x = 0$ in $f = 0$. Then we get $y^3z + y^4 = 0$, which implies $y = 0$ or $y + z = 0$. Therefore there are two points in $X \backslash X_0$, given by $p_1 = [0 : 0 : 1]$ and $p_2 = [0 : -1 : 1]$ respectively. To check whether they are singular points, we need to find a standard affine piece which contain them. Since the z-coordinates of p_1 and p_2 are non-zero, we can choose $X_2 = X \cap U_2$. The standard affine piece $X_2 = \mathbb{V}(f_2)$ where $f_2 = x^3 + x^2y + y^3 + x^4 + y^4$. The non-homogeneous coordinates of p_1 and p_2 are given by $p_1 = (0, 0)$ and $p_2 = (0, -1)$

respectively. The partial derivatives of f_2 are

$$\frac{\partial f_2}{\partial x} = 3x^2 + 2xy + 4x^3;$$
$$\frac{\partial f_2}{\partial y} = x^2 + 3y^2 + 4y^3.$$

It is easy to see that at the point $p_1 = (0,0)$, we have $f_2(p_1) = \frac{\partial f_2}{\partial x}(p_1) = \frac{\partial f_2}{\partial y}(p_1) = 0$. Therefore p_1 is a singular point on X_2. At the point $p_2 = (0,-1)$, we have $\frac{\partial f_2}{\partial y}(p_2) = -1 \neq 0$. Therefore p_2 is a non-singular point on X_2. By Definition 7.8, the only singular point of X is $p_1 = [0:0:1]$.

Solution 7.3. *Example: plane cubics.* There are three cases to deal with in this question. Most of the calculations are the same in all the three cases. First of all we look at a standard affine piece of $X = \mathbb{V}(f) \subseteq \mathbb{P}^2$. You can choose any standard affine piece of X to start with. For example, we choose the standard affine pice $X_2 = X \cap U_2$, which is given by setting $z = 1$ in f. Therefore we have

$$X_2 = \mathbb{V}(y^2 - (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)) \subseteq \mathbb{A}^2.$$

To find the singular points on X_2, we need to solve the system

$$\begin{aligned} y^2 - (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) &= 0; \\ -(x-\lambda_2)(x-\lambda_3) - (x-\lambda_1)(x-\lambda_3) - (x-\lambda_1)(x-\lambda_2) &= 0; \\ 2y &= 0. \end{aligned}$$

The third equation implies $y = 0$, then the first equation implies $x = \lambda_1$ or λ_2 or λ_3. Now there is some difference in the three cases.

(1) If λ_1, λ_2 and λ_3 are distinct, then it is clear that none of them is a solution to the second equation. Therefore X_2 is non-singular in this case.

(2) If two of the three are equal, say, $\lambda_1 = \lambda_2 \neq \lambda_3$, then it is clear that $x = \lambda_1$ (or λ_2) is a solution to the second equation while $x = \lambda_3$ is not a solution. Therefore X_2 has a singular point $(\lambda_1, 0)$, which has homogeneous coordinates $[\lambda_1 : 0 : 1]$ as a point in X.

(3) If all the three are equal, then $x = \lambda_1$ (or λ_2 or λ_3) is a solution to the second equation. Therefore X_2 has a singular point $(\lambda_1, 0)$, which has homogeneous coordinates $[\lambda_1 : 0 : 1]$ as a point in X.

It remains to consider the points in $X \backslash X_2$. To find these points we set $z = 0$ in the equation $f = 0$. We get $-x^3 = 0$ hence $x = 0$. Therefore the only point in $X \backslash X_2$ is $p = [x : y : z] = [0 : 1 : 0]$. Since the y-coordinate of p is non-zero, it is a point in the standard affine piece $X_1 = X \cap U_1$, given by the non-homogeneous coordinates $p = (0,0)$.

To write down the defining polynomial for X_1 we set $y = 1$ and get $X_1 = \mathbb{V}(f_1) \subseteq \mathbb{A}^2$ where

$$f_1 = z - (x - \lambda_1 z)(x - \lambda_2 z)(x - \lambda_3 z).$$

Its partial derivative with respect to z is given by

$$\frac{\partial f_1}{\partial z} = 1 + \lambda_1(x - \lambda_2 z)(x - \lambda_3 z) + \lambda_2(x - \lambda_1 z)(x - \lambda_3 z) + \lambda_3(x - \lambda_1 z)(x - \lambda_2 z).$$

It is clear that at the point $p = (0, 0)$, we have $\frac{\partial f_1}{\partial z}(p) = 1 \neq 0$. Therefore $p = (0, 0)$ is a non-singular point of X_1, hence $p = [0 : 1 : 0]$ is a non-singular point of X. This holds in all the three cases. We have the following conclusion:

(1) If λ_1, λ_2 and λ_3 are distinct, X is non-singular.

(2) If two of the three are equal, say, $\lambda_1 = \lambda_2 \neq \lambda_3$, then X has a unique singular point $[\lambda_1 : 0 : 1]$.

(3) If all the three are equal, then X has a unique singular point $[\lambda_1 : 0 : 1]$.

Solution 7.4. *Example: projective twisted cubic.* We first consider the standard affine piece $Y_0 = Y \cap U_0$. By settin $z_0 = 1$ we get

$$Y_0 = \mathbb{V}_a(y_2 - y_1^2, y_1 y_3 - y_2^2, y_3 - y_1 y_2).$$

To find the dimension of the tangent space at any point $p = (y_1, y_2, y_3)$, we consider the matrix of partial derivatives:

$$M_p = \begin{pmatrix} -2y_1 & 1 & 0 \\ y_3 & -2y_2 & y_1 \\ -y_2 & -y_1 & 1 \end{pmatrix}.$$

We need to find $\operatorname{rank} M_p$. First we compute the determinant of M_p:

$$\det M_p = 4y_1 y_2 - y_1 y_2 - y_3 - 2y_1^3 = 4y_1 y_2 - y_1 y_2 - y_1 y_2 - 2y_1 y_2 = 0.$$

Therefore $\operatorname{rank} M_p \leqslant 2$. Notice that the first and third rows of M_p are linearly independent (or the second and third columns). Therefore $\operatorname{rank} M_p = 2$, which implies $\dim T_p Y_0 = 1$ at every $p \in Y_0$. It follows that Y_0 is non-singular and $\dim Y = \dim Y_0 = 1$.

Now we consider the points in $Y \backslash Y_0$. Let $p = [y_0 : y_1 : y_2 : y_3]$ be such a point, then $y_0 = 0$, which implies $y_1^2 = y_0 y_2 = 0$ and $y_2^2 = y_1 y_3 = 0$. Therefore the only point $p \in Y \backslash Y_0$ is given by $p = [0 : 0 : 0 : 1]$. To determine whether p is a singular point, we need to look at the standard affine piece $Y_3 = Y \cap U_3$. We could perform a similar calculation as above to show that Y_3 is non-singular. More precisely, we have

$$Y_3 = \mathbb{V}_a(y_0 y_2 - y_1^2, y_1 - y_2^2, y_0 - y_1 y_2).$$

For any point $q = (y_0, y_1, y_2) \in Y_3$, the matrix

$$M_q = \begin{pmatrix} y_2 & -2y_1 & y_0 \\ 0 & 1 & -2y_2 \\ 1 & -y_2 & -y_1 \end{pmatrix}.$$

We notice that

$$\det M_q = -y_1y_2 + 4y_1y_2 - y_0 - 2y_2^3 = -y_1y_2 + 4y_1y_2 - y_1y_2 - 2y_1y_2 = 0.$$

Therefore $\operatorname{rank} M_q \leqslant 2$. Moreover the second and the third rows are linearly independent, hence $\operatorname{rank} M_q = 2$ for every $q \in Y_3$. It follows that Y_3 is non-singular. To summarise, Y is non-singular and has dimension 1.