Solutions to Exercise Sheet 8

Solution 8.1. *Examples of rational curves.*

(1) We claim that $\varphi : \mathbb{P}^1 \longrightarrow L;\ [x : y] \longmapsto [x : y : 0]$ is a morphism. It is given by homogeneous polynomials of the same degree, and is everywhere defined, since x and y cannot be both zero. The image of any point under φ lies in L because the last coordinate is zero. This justifies the claim. Similarly we claim that $\psi : L \longrightarrow \mathbb{P}^1;\ [x : y : z] \longmapsto [x : y]$ is a morphism. It is given by homogeneous polynomials of the same degree. Since $z = 0$, x and y cannot be both zero, hence it is defined for every point in L. The image of any point in L under φ is clearly in $\mathbb{P}^1$. This justifies the claim. Finally we check φ and ψ are inverse to each other. For any point $[x : y] \in \mathbb{P}^1$, $(\psi \circ \varphi)([x : y]) = \psi([x : y : 0]) = [x : y]$. For any point $[x : y : z] \in L$, $(\varphi \circ \psi)([x : y : z]) = \varphi([x : y]) = [x : y : 0] = [x : y : z]$ since $z = 0$. Therefore L is isomorphic to $\mathbb{P}^1$. In particular, they are birational, hence L is rational.

(2) Define rational maps $\varphi_2 : \mathbb{P}^1 \dashrightarrow C_2$ by $\varphi_2([u : v]) = [uv^2 : v^3 : u^3]$ and $\psi_2 : C_2 \dashrightarrow \mathbb{P}^1$ by $\psi_2([x : y : z]) = [x : y]$. To show φ_2 is a rational map, we observe: all components are homogeneous of degree 3; φ_2 is defined at every point $[u : v] \in \mathbb{P}^1$ since either u^3 or v^3 is non-zero; the image $[uv^2 : v^3 : u^3]$ is a point in C_2 since it satisfies the defining equation of C_2. To show ψ_2 is a rational map, we observe: all components are homogeneous of degree 1; ψ_2 is well-defined at every point on C_2 except $[0 : 0 : 1]$; image of ψ_2 is clearly in $\mathbb{P}^1$. It remains to show φ_2 and ψ_2 are mutually inverse to each other. For every $[u : v] \in \mathbb{P}^1$ where $\psi_2 \circ \varphi_2$ is defined, we have $(\psi_2 \circ \varphi_2)([u : v]) = \psi_2([uv^2 : v^3 : u^3]) = [uv^2 : v^3] = [u : v]$. For every $[x : y : z] \in C$ where $\varphi_2 \circ \psi_2$ is defined, we have $(\varphi_2 \circ \psi_2)([x : y : z]) = \varphi_2([x : y]) = [xy^2 : y^3 : x^3] = [xy^2 : y^3 : y^2 z] = [x : y : z]$. Therefore C_2 is birational to $\mathbb{P}^1$, hence is rational.

Solution 8.2. *Example: Fermat cubic.*

(1) We consider the standard affine piece $C_0 = C \cap U_0 = \mathbb{V}_a(f_0) \subseteq \mathbb{A}^2$ where $f_0 = 1 + y^3 + z^3$. Since $\frac{\partial f_0}{\partial y} = 3y^2$ and $\frac{\partial f_0}{\partial z} = 3z^2$, the two derivatives vanish if and only if $y = z = 0$. But then $f_0 = 1 \neq 0$. Therefore $f_0 = \frac{\partial f_0}{\partial y} = \frac{\partial f_0}{\partial z} = 0$ have no common solution, which means C_0 is non-singular. Since the equation of C is symmetric with respect to the variables, the same calculation shows that all other standard affine pieces are also non-singular. Therefore C is non-singular.

(2) A point on the line L can be given by $p = [x : y : 0]$. If $p \in C$, then we have $x^3 + y^3 = 0$, hence $y = -x$ or $-\omega x$ or $-\omega^2 x$ where $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$ is a primitive third root of unity. So the three points in $L \cap C$ are $p_1 = [1 : -1 : 0]$, $p_2 = [1 : -\omega : 0]$ and $p_3 = [1 : -\omega^2 : 0]$.

(3) At least one of the three coordinates is non-zero. Without loss of generality, we can assume $a \neq 0$. Then the point $p = [a : b : c] \in C_0 = C \cap U_0 = \mathbb{V}_a(f_0) \subseteq \mathbb{A}^2$, in which its non-homogeneous coordinates are given by $p = (\frac{b}{a}, \frac{c}{a})$. The tangent space of p in the standard affine piece C_0 is given by

$$T_pC_0 = \mathbb{V}_a\left(3 \cdot \frac{b^2}{a^2} \cdot (y - \frac{b}{a}) + 3 \cdot \frac{c^2}{a^2} \cdot (z - \frac{c}{a})\right).$$

The tangent space T_pC is the projective closure of T_pC_0, which is given by the homogenisation of the above polynomial

$$T_pC = \mathbb{V}_p\left(3 \cdot \frac{b^2}{a^2} \cdot (y - \frac{b}{a}x) + 3 \cdot \frac{c^2}{a^2} \cdot (z - \frac{c}{a}x)\right).$$

Since we assumed $a \neq 0$, we can multiply this polynomial by $\frac{a^3}{3}$ without changing its vanishing locus. Then we get

$$\begin{aligned} T_pC &= \mathbb{V}_p(b^2(ay - bx) + c^2(az - cx)) \\ &= \mathbb{V}_p((-b^3 - c^3)x + ab^2y + ac^2z) \\ &= \mathbb{V}_p(a^3x + ab^2y + ac^2z) \\ &= \mathbb{V}_p(a^2x + b^2y + c^2z). \end{aligned}$$

In the last step above is valid since we assumed $a \neq 0$.

Since a, b and c are symmetric, a similar calculation will give the same equation for the tangent space T_pC when $b \neq 0$ or $c \neq 0$.

(4) At the point $p_1 = [1 : -1 : 0]$, the tangent space $T_{p_1}C = \mathbb{V}_p(x + y)$. For any point $q = [x : y : z] \in T_{p_1}C$, we have $x = -y$. If $q \in C$, we then have $(-y)^3 + y^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution with multiplicity 3. This means $T_{p_1}C$ meet C at one point with multiplicity 3, hence p_1 is an inflection point.

Similarly, at the point $p_2 = [1 : -\omega : 0]$, the tangent space $T_{p_2}C = \mathbb{V}_p(x + \omega^2 y)$. For any point $q = [x : y : z] \in T_{p_2}C$, we have $x = -\omega^2 y$. If $q \in C$, we then have $(-\omega^2 y)^3 + y^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution with multiplicity 3. This means $T_{p_2}C$ meet C at one point with multiplicity 3, hence p_2 is an inflection point.

Moreover, at the point $p_3 = [1 : -\omega^2 : 0]$, the tangent space $T_{p_3}C = \mathbb{V}_p(x + \omega y)$. For any point $q = [x : y : z] \in T_{p_3}C$, we have $x = -\omega y$. If $q \in C$, we then have $(-\omega y)^3 + y^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution with multiplicity 3. This means $T_{p_3}C$ meet C at one point with multiplicity 3, hence p_3 is an inflection point.

Solution 8.3. *Bézout's theorem for conics.*

(1) If $C = L_1 \cup L_2$, then every common point of C and D must be either a common point of L_1 and D, or a common point of L_2 and D. We know by Theorem

8.8 that $L_1 \cap D$ comprises at most d points, or precisely d points when counting with multiplicities; $L_2 \cap D$ comprises at most d points, or precisely d points when counting with multiplicities. Therefore $C \cap D$ comprises at most $2d$ points, or precisely $2d$ points when counting with multiplicities.

(2) We have proved in Example 5.23 that C is isomorphic to $\mathbb{P}^1$. In particular, every point in C can be given by $[p^2 : pq : q^2]$ for some $[p : q] \in \mathbb{P}^1$. Let $D = \mathbb{V}(f)$ for some homogeneous polynomial $f(x, y, z)$ of degree d. Then $[p^2 : pq : q^2] \in \mathbb{V}(f)$ if and only if $f(p^2, pq, q^2) = 0$. The left-hand side is a homogeneous polynomial of degree $2d$ in p and q. By Exercise 4.4 (2), it can be completely factored into $2d$ homogeneous factors of degree 1 as

$$f(p^2, pq, q^2) = (a_1 p + b_1 q) \cdots (a_{2d} p + b_{2d} q) = 0.$$

Each factor $a_i p + b_i q$ determines a point $[p : q] = [b_i : -a_i] \in \mathbb{P}^1$, hence $f = 0$ has at most $2d$ solutions $[p : q] = [b_i : -a_i] \in \mathbb{P}^1$, which give at most $2d$ points $[p^2 : pq : q^2] = [b_i^2 : -a_i b_i : a_i^2] \in (C \cap D)$. When counting the number of times each point occurs as a solution, we get precisely $2d$ points.

Solution 8.4. *An interesting application of Bézout's theorem.*

(1) By Example 8.3, every conic C is given by a homogeneous polynomial $g(x, y, z) = 0$ of degree 2 with 6 coefficients a, b, c, d, e and f. For each i, since $p_i = [x_i : y_i : z_i] \in C$, we can plug in $x = x_i$, $y = y_i$ and $z = z_i$ to get an equation $g(x_i, y_i, z_i) = 0$, which is a homogeneous linear equation in a, b, c, d, e and f. In this way the 5 points give a system of 5 linear equations. Since there are 5 equations and 6 indeterminants, by the theorem of rank-nullity, there is a solution for a, b, c, d, e and f such that they are not simultaneously zero. This solution determines the homogeneous polynomial $g(x, y, z)$ of degree 2. We claim that g has no repeated factors. If g has repeated factors, then g is the square of a linear polynomial hence gives a double line which passes through all the 5 given points. This is a contradiction since no 4 of the given points are allowed to be on the same line. Hence we conclude that g defines a conic.

(2) Assume that there are two distinct conics C_1 and C_2, both of which pass through the 5 points. By Theorem 8.12, if they do not have any common component, then they can meet in at most 4 common points. Hence they must have a common component.

(3) If either C_1 or C_2 is an irreducible conic, which has only one component, then the other must be the same conic. Under the assumption that C_1 and C_2 are distinct conics, both of them must be the unions of two lines. Since they have a common component, the other component in the two conics must be distinct. Hence we can assume $C_1 = L_0 \cup L_1$ and $C_2 = L_0 \cup L_2$, where L_0, L_1 and L_2 are distinct lines.

We know the 5 points $p_1, \cdots, p_5$ are on both conics. For each p_i, there are two possibilities: $p_i \in L_0$, or $p_i \notin L_0$. If the second possibility happens, then $p_i \in L_1$ since $p_i \in C_1$, and $p_i \in L_2$ since $p_i \in C_2$. This implies p_i is a common point of L_1 and L_2. Since L_1 and L_2 are distinct lines, by Theorem 8.8, they have only 1 common point. It follows that among the 5 points $p_1, \cdots, p_5$, at most one of them is not on L_0; in other words, at least 4 of them are on the line L_0. This is a contradiction because no 4 of them are allowed to be on the same line.