Solutions to Exercise Sheet 8

Solution 8.1. Examples of rational curves.

(1) We claim that \( \varphi : \mathbb{P}^1 \to L; [x : y] \mapsto [x : y : 0] \) is a morphism. It is given by homogeneous polynomials of the same degree, and is everywhere defined, since \( x \) and \( y \) cannot be both zero. The image of any point under \( \varphi \) lies in \( L \) because the last coordinate is zero. This justifies the claim. Similarly we claim that \( \psi : L \to \mathbb{P}^1; [x : y : z] \mapsto [x : y] \) is a morphism. It is given by homogeneous polynomials of the same degree. Since \( z = 0 \), \( x \) and \( y \) cannot be both zero, hence it is defined for every point in \( L \). The image of any point in \( L \) under \( \varphi \) is clearly in \( \mathbb{P}^1 \). This justifies the claim. Finally we check \( \varphi \) and \( \psi \) are inverse to each other. For any point \([x : y] \in \mathbb{P}^1\), \((\psi \circ \varphi)([x : y]) = \psi([x : y : 0]) = [x : y]\). For any point \([x : y : z] \in L\), \((\varphi \circ \psi)([x : y : z]) = \varphi([x : y]) = [x : y : 0] = [x : y : z]\) since \( z = 0 \). Therefore \( L \) is isomorphic to \( \mathbb{P}^1 \). In particular, they are birational, hence \( L \) is rational.

(2) Define rational maps \( \varphi_2 : \mathbb{P}^1 \to C_2 \) by \( \varphi_2([u : v]) = [uv^2 : v^3 : u^3] \) and \( \psi_2 : C_2 \to \mathbb{P}^1 \) by \( \psi_2([x : y : z]) = [x : y] \). To show \( \varphi_2 \) is a rational map, we observe: all components are homogeneous of degree 3; \( \varphi_2 \) is defined at every point \([u : v] \in \mathbb{P}^1\) since either \( u^3 \) or \( v^3 \) is non-zero; the image \([uv^2 : v^3 : u^3]\) is a point in \( C_2 \) since it satisfies the defining equation of \( C_2 \). To show \( \psi_2 \) is a rational map, we observe: all components are homogeneous of degree 1; \( \psi_2 \) is well-defined at every point on \( C_2 \) except \([0 : 0 : 1]\); image of \( \psi_2 \) is clearly in \( \mathbb{P}^1 \). It remains to show \( \varphi_2 \) and \( \psi_2 \) are mutually inverse to each other. For every \([u : v] \in \mathbb{P}^1\) where \( \psi_2 \circ \varphi_2 \) is defined, we have \((\psi_2 \circ \varphi_2)([u : v]) = \psi_2([uv^2 : v^3 : u^3]) = [uv^2 : v^3] = [u : v]\). For every \([x : y : z] \in C_2\) where \( \varphi_2 \circ \psi_2 \) is defined, we have \((\varphi_2 \circ \psi_2)([x : y : z]) = \varphi_2([x : y]) = [xy^2 : y^3 : x^3] = [xy^2 : y^3 : y^2 z] = [x : y : z]\). Therefore \( C_2 \) is birational to \( \mathbb{P}^1 \), hence is rational.

Solution 8.2. Example: Fermat cubic.

(1) We consider the standard affine piece \( C_0 = C \cap U_0 = V_a(f_0) \subseteq \mathbb{A}^2 \) where \( f_0 = 1 + y^3 + z^3 \). Since \( \frac{\partial f_0}{\partial y} = 3y^2 \) and \( \frac{\partial f_0}{\partial z} = 3z^2 \), the two derivatives vanish if and only if \( y = z = 0 \). But then \( f_0 = 1 \neq 0 \). Therefore \( f_0 = \frac{\partial f_0}{\partial y} = \frac{\partial f_0}{\partial z} = 0 \) have no common solution, which means \( C_0 \) is non-singular. Since the equation of \( C \) is symmetric with respect to the variables, the same calculation shows that all other standard affine pieces are also non-singular. Therefore \( C \) is non-singular.

(2) A point on the line \( L \) can be given by \( p = [x : y : 0] \). If \( p \in C \), then we have \( x^3 + y^3 = 0 \), hence \( y = -x \) or \( -\omega x \) or \( -\omega^2 x \) where \( \omega = e^{\frac{2\pi i}{3}} \) is a primitive third root of unity. So the three points in \( L \cap C \) are \( p_1 = [1 : -1 : 0], p_2 = [1 : -\omega : 0] \) and \( p_3 = [1 : -\omega^2 : 0] \).
(3) At least one of the three coordinates is non-zero. Without loss of generality, we can assume $a \neq 0$. Then the point $p = [a : b : c] \in C_0 = C \cap U_0 = \mathbb{V}_a(f_0) \subseteq \mathbb{A}^2$, in which its non-homogeneous coordinates are given by $p = (\frac{b}{a}, \frac{c}{a})$. The tangent space of $p$ in the standard affine piece $C_0$ is given by

$$T_pC_0 = \mathbb{V}_a \left( 3 \cdot \frac{b^2}{a^2} \cdot (y - \frac{b}{a}) + 3 \cdot \frac{c^2}{a^2} \cdot (z - \frac{c}{a}) \right).$$

The tangent space $T_pC$ is the projective closure of $T_pC_0$, which is given by the homogenisation of the above polynomial

$$T_pC = \mathbb{V}_p \left( 3 \cdot \frac{b^2}{a^2} \cdot (y - \frac{b}{a} x) + 3 \cdot \frac{c^2}{a^2} \cdot (z - \frac{c}{a} x) \right).$$

Since we assumed $a \neq 0$, we can multiply this polynomial by $\frac{a^3}{3}$ without changing its vanishing locus. Then we get

$$T_pC = \mathbb{V}_p \left( b^2 (ay - bx) + c^2 (az - cx) \right)$$

$$= \mathbb{V}_p \left( (-b^3 - c^3)x + ab^2 y + ac^2 z \right)$$

$$= \mathbb{V}_p \left( a^2 x + b^2 y + c^2 z \right).$$

In the last step above is valid since we assumed $a \neq 0$.

Since $a$, $b$ and $c$ are symmetric, a similar calculation will give the same equation for the tangent space $T_bC$ when $b \neq 0$ or $c \neq 0$.

(4) At the point $p_1 = [1 : -1 : 0]$, the tangent space $T_{p_1}C = \mathbb{V}_p(x + y)$. For any point $q = [x : y : z] \in T_{p_1}C$, we have $x = -y$. If $q \in C$, we then have $(-y)^3 + y^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution with multiplicity 3. This means $T_{p_1}C$ meet $C$ at one point with multiplicity 3, hence $p_1$ is an inflection point.

Similarly, at the point $p_2 = [1 : \omega : 0]$, the tangent space $T_{p_2}C = \mathbb{V}_p(x + \omega^2 y)$. For any point $q = [x : y : z] \in T_{p_2}C$, we have $x = -\omega^2 y$. If $q \in C$, we then have $(-\omega^2 y)^3 + y^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution with multiplicity 3. This means $T_{p_2}C$ meet $C$ at one point with multiplicity 3, hence $p_2$ is an inflection point.

Moreover, at the point $p_3 = [1 : -\omega^2 : 0]$, the tangent space $T_{p_3}C = \mathbb{V}_p(x + \omega y)$. For any point $q = [x : y : z] \in T_{p_3}C$, we have $x = -\omega y$. If $q \in C$, we then have $(-\omega y)^3 + y^3 + z^3 = 0$ hence $z^3 = 0$, which has one solution with multiplicity 3. This means $T_{p_3}C$ meet $C$ at one point with multiplicity 3, hence $p_3$ is an inflection point.

---

**Solution 8.3. Bézout’s theorem for conics.**

(1) If $C = L_1 \cup L_2$, then every common point of $C$ and $D$ must be either a common point of $L_1$ and $D$, or a common point of $L_2$ and $D$. We know by Theorem
8.8 that \( L_1 \cap D \) comprises at most \( d \) points, or precisely \( d \) points when counting with multiplicities; \( L_2 \cap D \) comprises at most \( d \) points, or precisely \( d \) points when counting with multiplicities. Therefore \( C \cap D \) comprises at most \( 2d \) points, or precisely \( 2d \) points when counting with multiplicities.

(2) We have proved in Example 5.23 that \( C \) is isomorphic to \( \mathbb{P}^1 \). In particular, every point in \( C \) can be given by \([p^2 : pq : q^2]\) for some \([p : q] \in \mathbb{P}^1\). Let \( D = \mathbb{V}(f) \) for some homogeneous polynomial \( f(x, y, z) \) of degree \( d \). Then \([p^2 : pq : q^2] \in \mathbb{V}(f)\) if and only if \( f(p^2, pq, q^2) = 0\). The left-hand side is a homogeneous polynomial of degree \( 2d \) in \( p \) and \( q \). By Exercise 4.4 (2), it can be completely factored into \( 2d \) homogeneous factors of degree 1 as

\[
f(p^2, pq, q^2) = (a_1p + b_1q) \cdots (a_{2d}p + b_{2d}q) = 0.
\]

Each factor \( a_ip + b_iq \) determines a point \([p : q] = [b_i : -a_i] \in \mathbb{P}^1\), hence \( f = 0 \) has at most \( 2d \) solutions \([p : q] = [b_i : -a_i] \in \mathbb{P}^1\), which give at most \( 2d \) points \([p^2 : pq : q^2] = [b_i^2 : -a_ib_i : a_i^2] \in (C \cap D)\). When counting the number of times each point occurs as a solution, we get precisely \( 2d \) points.

**Solution 8.4.** An interesting application of Bézout’s theorem.

(1) By Example 8.3, every conic \( C \) is given by a homogeneous polynomial \( g(x, y, z) = 0 \) of degree 2 with 6 coefficients \( a, b, c, d, e \) and \( f \). For each \( i \), since \( p_i = [x_i : y_i : z_i] \in C \), we can plug in \( x = x_i \), \( y = y_i \) and \( z = z_i \) to get an equation \( g(x_i, y_i, z_i) = 0 \), which is a homogeneous linear equation in \( a, b, c, d, e \) and \( f \). In this way the 5 points give a system of 5 linear equations. Since there are 5 equations and 6 indeterminants, by the theorem of rank-nullity, there is a solution for \( a, b, c, d, e \) and \( f \) such that they are not simultaneously zero. This solution determines the homogeneous polynomial \( g(x, y, z) \) of degree 2. We claim that \( g \) has no repeated factors. If \( g \) has repeated factors, then \( g \) is the square of a linear polynomial hence gives a double line which passes through all the 5 given points. This is a contradiction since no 4 of the given points are allowed to be on the same line. Hence we conclude that \( g \) defines a conic.

(2) Assume that there are two distinct conics \( C_1 \) and \( C_2 \), both of which pass through the 5 points. By Theorem 8.12, if they do not have any common component, then they can meet in at most 4 common points. Hence they must have a common component.

(3) If either \( C_1 \) or \( C_2 \) is an irreducible conic, which has only one component, then the other must be the same conic. Under the assumption that \( C_1 \) and \( C_2 \) are distinct conics, both of them must be the unions of two lines. Since they have a common component, the other component in the two conics must be distinct. Hence we can assume \( C_1 = L_0 \cup L_1 \) and \( C_2 = L_0 \cup L_2 \), where \( L_0, L_1 \) and \( L_2 \) are distinct lines.
We know the 5 points \( p_1, \cdots, p_5 \) are on both conics. For each \( p_i \), there are two possibilities: \( p_i \in L_0 \), or \( p_i \notin L_0 \). If the second possibility happens, then \( p_i \in L_1 \) since \( p_i \in C_1 \), and \( p_i \in L_2 \) since \( p_i \in C_2 \). This implies \( p_i \) is a common point of \( L_1 \) and \( L_2 \). Since \( L_1 \) and \( L_2 \) are distinct lines, by Theorem 8.8, they have only 1 common point. It follows that among the 5 points \( p_1, \cdots, p_5 \), at most one of them is not on \( L_0 \); in other words, at least 4 of them are on the line \( L_0 \). This is a contradiction because no 4 of them are allowed to be on the same line.