Solution 9.1. *Understanding the simplified group law.*

(1) We show that the point $p = [0:1:0]$ is an inflection point on the non-singular cubic $C = \mathbb{V}_p(f)$ where $f = y^2z - x^3 - ax^2z - bxz^2 - cz^3$. First of all we need to find out the tangent line T_pC, which can be computed on the standard affine piece $C_1 = C \cap U_1 = \mathbb{V}_a(f_1)$ where $f_1 = z - x^3 - ax^2z - bxz^2 - cz^3$. The non-homogeneous coordinates of p in U_1 is $p = (0,0)$. Since $\frac{\partial f_1}{\partial x} = -3x^2 - 2axz - bz^2$ and $\frac{\partial f_1}{\partial z} = 1 - ax^2 - 2bxz - 3cz^2$, the tangent line $T_pC_1 = \mathbb{V}_a(0(x-0) + 1(z-0)) = \mathbb{V}_a(z)$. Its projective closure is $T_pC = \mathbb{V}_p(z)$. To find the intersection points of C and T_pC, we follow the method in the proof of Theorem 8.8. A point on T_pC is given by $[x:y:0]$. It lies in C if and only if $f(x,y,0) = 0$, where $f(x,y,0) = -x^3$ which has one solution $[x:y] = [0:1]$ with multiplicity 3. Therefore T_pC and C meet at the point $[0:1:0]$ with multiplicity 3, which proves $p = [0:1:0]$ is an inflection point on C.

(2) First of all, since O is the identity element in the group law, we always have $O + O = O$, so O is one of such point. It remains to find all such points $P \in C_2$. The condition $P + P = O$ can be interpreted as $P = -P$. If the non-homogeneous coordinates of P in C_2 is given by $P = (x,y)$, then by the simplified group law 9.4, $-P = (x, -y)$. The condition $P = -P$ holds if and only if $y = 0$. Therefore all points $P \in C$ satisfying $P + P = O$ are precisely the identity element $O = [0:1:0]$ and those points $P = (x,y) \in C_2$ such that $y = 0$.

(3) In the standard affine piece $C_2 = \mathbb{V}(y^2 - x^3 + 4x - 1)$, the non-homogeneous coordinates of the two points are $A = (2,1)$ and $B = (-2,-1)$. The line AB is given by $x - 2y = 0$. To find its third intersection points with C_2, we need to solve the system

$$y^2 - x^3 + 4x - 1 = 0,$$
$$x - 2y = 0.$$

We substitute x by $2y$ in the first equation to get $y^2 - 8y^3 + 8y - 1 = 0$, which can be factored as $(y^2 - 1)(1 - 8y) = 0$. The solutions are $y = \pm 1$ and $y = \frac{1}{8}$. Therefore the third intersection point is $(\frac{1}{4}, \frac{1}{8})$, whose reflection across the x-axis is the sum of A and B; that is $A + B = (\frac{1}{4}, -\frac{1}{8})$, or $[\frac{1}{4} : -\frac{1}{8} : 1]$ in homogeneous coordinates (or $[2 : -1 : 8]$ if you prefer). The inverse $-A$ is the reflection of A across the x-axis, so $-A = (2,-1)$, or $[2 : -1 : 1]$ in homogeneous coordinates. The inverse $-B$ is the reflection of B across the x-axis, so $-B = (-2, 1)$, or $[-2 : 1 : 1]$ in homogeneous coordinates.

Solution 9.2. *Example of group law.*

(1) For $L_1 \cap C$, set $z = 0$ in the equation defining C to obtain $x^3 = 0$, which gives solutions $[x : y] = [0 : 1]$ with multiplicity 3. Hence $[x : y : z] = [0 : 1 : 0]$ is the only intersection point with multiplicity 3. For $L_2 \cap C$, set $x = 0$ in the equation defining C to obtain $y^2 z = 0$, which gives solutions $[y : z] = [0 : 1]$ with multiplicity 2 and $[1 : 0]$ with multiplicity 1. Hence the line L_2 meets C at $[0 : 0 : 1]$ with multiplicity 2 and $[0 : 1 : 0]$ with multiplicity 1. For $L_3 \cap C$, set $y = 2x$ to obtain $x(4xz - x^2 - 4z^2) = 0$, which can be written as $-x(x - 2z)^2 = 0$. Its solutions are $[x : z] = [0 : 1]$ with multiplicity 1, and $[x : z] = [2 : 1]$ with multiplicity 2. Therefore L_3 meets C at $[x : y : z] = [0 : 0 : 1]$ with multiplicity 1 and $[2 : 4 : 1]$ with mulplicity 2.

(2) We can use the simplified group law 9.4. The standard affine piece $C_2 = \mathbb{V}_a(f_2) \subseteq \mathbb{A}^2$ where $f_2 = y^2 - x^3 - 4x$. We first compute $P + P$. The non-homogeneous coordinates of P are $(2, 4)$. To compute the tangent line $T_P C_2$, we find $\frac{\partial f_2}{\partial x} = -3x^2 - 4$ and $\frac{\partial f_2}{\partial y} = 2y$. Therefore $\frac{\partial f_2}{\partial x}(P) = -16$ and $\frac{\partial f_2}{\partial y} = 8$. It follows that $T_P C_2 = \mathbb{V}_a(-16(x-2)+8(y-4)) = \mathbb{V}_a(-2(x-2)+(y-4)) = \mathbb{V}_a(-2x+y) \subseteq \mathbb{A}^2$. To find the third intersection point of $T_P C_2$ and C, we solve the system of equations

$$\begin{aligned} y^2 - x^3 - 4x &= 0, \\ -2x + y &= 0. \end{aligned}$$

We substitute y by $2x$ in the first equation to get $4x^2 - x^3 - 4x = 0$, which is $-x(x - 2)^2 = 0$. Therefore the system has a solution $(x, y) = (2, 4)$ with multiplicity 2 and a solution $(x, y) = (0, 0)$ with multiplicity 1. The solution $(2, 4)$ corresponds to the point P, hence the third intersection point is $R = (0, 0)$. The sum $P + P$ is the reflection $\overline{R}$ of R across the x-axis, which is still $(0, 0)$. Hence $P + P = \overline{R} = (0, 0) = R$.

Now we compute $R + R$. Since $R = (0, 0)$, by the simplified group law 9.4 (2a), we immediately have $R + R = O$. Therefore $P + P + P + P = O$. It follows that the order of P must divide 4, which can only be 1 or 2 or 4. Since $P \neq O$, the order of P is not 1. Since $P + P = R \neq O$, the order of P is not 2. Therefore the order of P is 4, which means the subgroup generated by P has order 4.

(3) To find all points $Q \in C$ such that $Q + Q = O$, we use Exercise 9.1 (2). First of all $O = [0 : 1 : 0]$ is such a point. It remains to find all points $Q = (x, y) \in C_2$ such that $y = 0$. In the equation $f_2 = y^2 - x^3 - 4x = 0$ we set $y = 0$. Then we have $-x^3 - 4x = -x(x^2 + 4) = 0$. Hence $x = 0$ or $2\sqrt{-1}$ or $-2\sqrt{-1}$. The corresponding points are $Q = (0, 0)$ or $(2\sqrt{-1}, 0)$ or $(-2\sqrt{-1}, 0)$. In summary, we found 4 points $Q \in C$ such that $Q + Q = O$, which are $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[2\sqrt{-1} : 0 : 1]$ and $[-2\sqrt{-1} : 0 : 1]$.

Solution 9.3. *Example: Tate's normal form.*

Notice that the defining polynomial of the cubic does not meet the conditions required for using the simplified group law. So we need to use the group law 9.1.

(1) To find the inverse, we use the method in the proof of Proposition 9.8. We need to find the third intersection point $\overline{O}$ of $T_O C$ and C, then find the third intersection point of $\overline{O}P$ and C, which is $-P$.

Since C is the projective closure of C_2, we can write down its defining equation as $C = \mathbb{V}_p(y^2z + sxyz - tyz^2 - x^3 + tx^2z) \subseteq \mathbb{P}^2$. It is easy to see that $O = [0:1:0]$ is the only point at infinity. To find the tangent line $T_O C$, we need to consider the standard affine piece $C_1 = C \cap U_1$ which contains the point O. We have $C_1 = \mathbb{V}_a(f_1) \subseteq \mathbb{A}^2$ where $f_1 = z + sxz - tz^2 - x^3 + tx^2z$ and $O = (0,0) \in C_1$. Since $\frac{\partial f_1}{\partial x} = sz - 3x^2 + 2txz$ and $\frac{\partial f_1}{\partial z} = 1 + sx - 2tz + tx^2$, we have $\frac{\partial f_1}{\partial x}(O) = 0$ and $\frac{\partial f_1}{\partial z}(O) = 1$, hence $T_O C_1 = \mathbb{V}_a(z) \subseteq \mathbb{A}^2$. Taking its projective closure, we get $T_O C = \mathbb{V}_p(z) \subseteq \mathbb{P}^2$. To find the intersection points of $T_O C$ and C, we consider an arbitrary point $[x:y:z] = [x:y:0] \in T_O C$. If this point is also in C, then we set $z = 0$ in the defining equation of C to get $-x^3 = 0$. Therefore $T_O C$ and C meet at the only point $[x:y:z] = [0:1:0]$ with multiplicity 3, which means that the third intersection point $\overline{O}$ of $T_O C$ and C is still $\overline{O} = O = [0:1:0]$.

To find $-P$, we need to write down the line $\overline{O}P$. We first make an observation. Since $P = (a,b) \in C_2$, its coordinates have to satisfy the defining polynomial of C_2, namely

$$b^2 + sab - tb - a^3 + ta^2 = 0,$$

or equivalently

$$-a^3 + ta^2 = -b(b + sa - t).$$

The homogeneous coordinates of P are given by $P = [a:b:1]$. By Lemma 9.2 the line is given by

$$\det \begin{pmatrix} x & 0 & a \\ y & 1 & b \\ z & 0 & 1 \end{pmatrix} = x - az = 0.$$

To find the third intersection point of $\overline{O}P$ and C, we consider an arbitrary point $[x:y:z] = [az:y:z] \in \overline{O}P$. Since this point is also in C, we get

$$y^2z + sayz^2 - tyz^2 - a^3z^3 + ta^2z^3 = 0.$$

Using the observation above, we get

$$y^2z + (sa - t)yz^2 - b(b + sa - t)z^3 = 0$$

which can be factored into

$$z(y - bz)(y + (b + sa - t)z) = 0.$$

The three solutions are $[y : z] = [1 : 0]$, $[b : 1]$ and $[-b - sa + t : 1]$. Since $x = az$, the three intersection points of $\overline{O}P$ and C are $[x : y : z] = [0 : 1 : 0]$, $[a : b : 1]$ and $[a : -b - sa + t : 1]$. The first two points are $\overline{O}$ and P, hence $-P = [a : -b - sa + t : 1]$. The non-homogeneous coordinates of $-P$ with respect to C_2 is $-P = (a, -b - sa + t)$.

(2) To compute $Q + Q$, we need to find the tangent line T_QC. We know that $Q \in C_2 = \mathbb{V}_a(f_2) \subseteq \mathbb{A}^2$ where $f_2 = y^2 + sxy - ty - x^3 + tx^2$. The partial derivatives are given by $\frac{\partial f_2}{\partial x} = sy - 3x^2 + 2tx$ and $\frac{\partial f_2}{\partial y} = 2y + sx - t$. At the point $Q = (0, 0)$, their values are $\frac{\partial f_2}{\partial x}(Q) = 0$ and $\frac{\partial f_2}{\partial y}(Q) = -t$. Since $t \neq 0$, we have $T_QC_2 = \mathbb{V}_a(-ty) = \mathbb{V}_a(y) \subseteq \mathbb{A}^2$, hence $T_QC = \mathbb{V}_p(y) \subseteq \mathbb{P}^2$. To find the third intersection point R of the line T_QC and C, we consider an arbitrary point $[x : y : z] = [x : 0 : z] \in T_QC$. When this point is also on C, we can set $y = 0$ in the defining equation of C to get $-x^3 + tx^2z = 0$. It has solutions $[x : z] = [0 : 1]$ with multiplicity 2 and $[t : 1]$ with multiplicity 1. Therefore the intersection points of T_QC and C are given by $[x : y : z] = [0 : 0 : 1]$ with multiplicity 2 and $[t : 0 : 1]$ with multiplicity 1. Hence third intersection point R of T_QC and C is $R = [t : 0 : 1]$.

It remains to find the third intersection point of OR and C, which is the sum $Q + Q$. Fortunately we have done the computation in part (1). Indeed, we have seen that, given a point $P = [a : b : 1] \in C$, the line $OP(= \overline{O}P)$ meets C at a third point $[a : -sa + t - b : 1]$. Let $a = t$ and $b = 0$, then OR meets C at a third point $[t : -st + t : 1]$, or in non-homogeneous coordinates $(t, -st + t)$. Therefore $Q + Q = (t, -st + t) = (t, t(1 - s))$.

Solution 9.4. *Pascal's mystic hexagon.*

(1) A picture has been given in the exercise class. You can also find the same picture in [Section 2.11, Reid, Undergraduate Algebraic Geometry].

(2) From the picture we can see that C_1 and C_2 meet at 9 distinct points, i.e.

$$C_1 \cap C_2 = \{A, B, C, D, E, F, P, Q, R\}.$$

Indeed, the first six points are distinct by the assumption. None of the last three points is on X (otherwise a certain line meets X in 3 points), so none of them can coincide with any of the first six points. The last three points must also be distinct (otherwise two certain lines meet each other in 2 points).

(3) By assumption, the cubic curve C_3 passes through 8 of the above 9 points with the point R being the only possible exception. By Lemma 9.12, R must be on C_3 as well. Therefore R is either on the conic X or the line PQ. We claim that R is not on X. Otherwise, the line BCR and the conic X meet at three distinct points B, C and R, which violates Bézout's theorem 8.8. Therefore R is on the line PQ, which means that the points P, Q, R are colinear.