Solutions to Exercise Sheet 10

Solution 10.1. Infinitely many lines on planes.

(1) Since $z_0$ and $az_1 + bz_2 + cz_3$ are both homogeneous polynomials of degree 1 and not proportional to each other, $L = \mathbb{V}(z_0, az_1 + bz_2 + cz_3)$ defines a line in $\mathbb{P}^2$. To show that the line $L$ is in $P$, we just need to observe that every point on $L$ satisfies the equation $z_0 = 0$, hence is a point in $P$.

(2) Let $L = \mathbb{V}(z_0, az_1 + bz_2 + cz_3)$ and $L' = \mathbb{V}(z_0, a'z_1 + b'z_2 + c'z_3)$ be two such lines, where $[a : b : c] \neq [a' : b' : c']$. If a point $p = [z_0 : z_1 : z_2 : z_3]$ is an intersection point of $L$ and $L'$, then its coordinates satisfy the system of equations

$$z_0 = 0;$$
$$az_1 + bz_2 + cz_3 = 0;$$
$$a'z_1 + b'z_2 + c'z_3 = 0.$$

The first equation fixes the $z_0$ coordinate. For the other coordinates, we look at the second and the third equations. We look at the coefficient matrix

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}.$$

Since $[a : b : c]$ and $[a' : b' : c']$ represent different points in $\mathbb{P}^2$, both rows are non-zero and linearly independent. Hence the matrix has rank 2. It follows that the null-space has dimension 1, which means that there is a unique solution for $[z_1 : z_2 : z_3]$ (up to scaling). Therefore there is a unique intersection point $[z_0 : z_1 : z_2 : z_3]$ for the lines $L$ and $L'$.

Solution 10.2. Infinitely many lines on non-singular quadric surfaces.

(1) It is clear that for every point $[a : b] \in \mathbb{P}^1$, the two polynomials $az_0 + bz_1$ and $az_2 + bz_3$ are non-zero and homogeneous of degree 1. They are not proportional to each other, so $\mathbb{V}(az_0 + bz_1, az_2 + bz_3)$ defines a line $L$ in $\mathbb{P}^2$. We still need to show that every point in $L$ is a point in $Q$. Since $[a : b] \in \mathbb{P}^1$, we have either $a \neq 0$ or $b \neq 0$. If $a \neq 0$, then a point $p = [z_0 : z_1 : z_2 : z_3] \in L$ satisfies $z_0 = -\frac{b}{a}z_1$ and $z_2 = -\frac{b}{a}z_3$. Then

$$z_0z_3 - z_1z_2 = \left(-\frac{b}{a}\right) \cdot z_1 \cdot z_3 - z_1 \cdot \left(-\frac{b}{a}\right) \cdot z_3 = 0.$$

Hence $p \in Q$. If $b \neq 0$, a similar calculation shows that every point $p \in L$ also satisfies the equation $z_0z_3 - z_1z_2 = 0$ hence is a point in $Q$. We conclude that $L$ is a line in $Q$. 

(2) Consider two lines \( L = \mathbb{V}(az_0 + bz_1, az_2 + bz_3) \) and \( L' = \mathbb{V}(a'z_0 + b'z_1, a'z_2 + b'z_3) \) where \([a : b]\) and \([a' : b']\) are two different points in \( \mathbb{P}^1 \). If the two lines have a common point \([z_0 : z_1 : z_2 : z_3]\), then the system of equations

\[
\begin{align*}
az_0 + bz_1 &= 0, \\
az_2 + bz_3 &= 0, \\
a'z_0 + b'z_1 &= 0, \\
a'z_2 + b'z_3 &= 0
\end{align*}
\]

must have a non-zero solution. However, the coefficient matrix for the first and the third equations is \( \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \). Since \([a : b]\) and \([a' : b']\) are two different points in \( \mathbb{P}^1 \), the two rows are both non-zero and linearly independent. Hence the matrix has rank 2, which means that the only solution to these two equations is \( z_0 = z_1 = 0 \). For the same reason the only solution to the second and fourth equations is \( z_2 = z_3 = 0 \). Since the system of four equations has only a zero solution, \( L \) and \( L' \) do not have any common point. In other words, they are disjoint.

(3) For any point \( p = [z_0 : z_1 : z_2 : z_3] \in Q \), we first show that \( p \) lies on a certain line \( L = \mathbb{V}(az_0 + bz_1, az_2 + bz_3) \). There are two cases. Case 1. If \( z_0 \) and \( z_1 \) are not simultaneously zero, then we choose \([a : b] = [z_1 : -z_0]\) for the line \( L \). We claim that \( p \in L \). Indeed, for such a choice of \([a : b]\) we have \( az_0 + bz_1 = z_1z_0 - z_0z_1 = 0 \) and \( az_2 + bz_3 = z_1z_2 - z_0z_3 = 0 \). The claim holds. Case 2. If \( z_0 \) and \( z_1 \) are both zero, then \( z_2 \) and \( z_3 \) are not simultaneously zero. We can choose \([a : b] = [z_3 : -z_2]\) for the line \( L \). A similar calculation shows that \( p \in L \). In both cases, the point \( p \) lies on a certain line \( L = \mathbb{V}(az_0 + bz_1, az_2 + bz_3) \) for a suitable choice of \([a : b]\).

It remains to prove that \( p \) lies on only one of such lines. This is clear because we have seen from part (2) that two such lines are always disjoint.

(4) For every \([a : b] \in \mathbb{P}^1 \), \( \mathbb{V}(az_0 + bz_2, az_1 + bz_3) \) also defines a line. These lines are pairwisely disjoint, and every point in \( Q \) lies on exactly one of them. The proof can be obtained simply by switching \( z_1 \) and \( z_2 \) in the proof for the above three parts.

Solution 10.3. Rationality of a cubic surface.
(1) We need to verify that every point in the image of $\psi$ satisfies the defining equation of $S$. Indeed, we have
\[
\begin{align*}
z_0^2z_1 + z_1^2z_2 + z_2^2z_3 + z_3^2z_0 &= -r^2t^2(r^2s + t^3) + s^2(r^2s + t^3)^2 + t^2s^2(r^2s + t^3)^2 - t^4(r^2s + t^3) + t^2(r^2s + t^3)^2 - rt(r^2s + t^3) \\
&= -(r^2t^2s + t^3) \cdot (rt + s^2) \cdot (r^2s + t^3) + (s^2t^2 + rt^3) \cdot (r^2s + t^3)^2 \cdot (rt + s^2) \\
&= -t^2(r^2s + t^3) \cdot (rt + s^2) \cdot (r^2s + t^3) + t^2(s^2 + rt) \cdot (r^2s + t^3)^2 \cdot (rt + s^2) \\
&= -t^2 \cdot (rt + s^2)^2 \cdot (r^2s + t^3)^2 + t^2 \cdot (r^2s + t^3)^2 \cdot (rt + s^2)^2 \\
&= 0.
\end{align*}
\]
Therefore the statement holds.

(2) Let $[z_0 : z_1 : z_2 : z_3]$ be a point in $S$. Then these coordinates satisfy
\[
z_0^2z_1 + z_1^2z_2 + z_2^2z_3 + z_3^2z_0 = 0.
\]
Then we have
\[
(\psi \circ \varphi)([z_0 : z_1 : z_2 : z_3]) = \psi([z_0z_3 : z_1z_2 : z_2z_3]) = [z_0z_2z_3^2(z_0z_2z_3^2 + z_1^2z_2^2) : -z_1z_2(z_0^2z_1z_2z_3^2 + z_3^2z_3^3) : \\
: z_2^2z_3^3(z_0z_2z_3 + z_1^2z_2^2) : -z_2z_3(z_0^2z_1z_2z_3 + z_3^2z_3^3)] = [z_0z_2z_3^2(z_3^2z_0 + z_1^2z_2^2) : -z_1z_2^2z_3(z_0^2z_1 + z_2^2z_3) : \\
: z_2^3z_3(z_3^2z_0 + z_1^2z_2) : -z_2^3z_3(z_0^2z_1 + z_2^2z_3)] = [z_0z_2z_3(z_3^2z_0 + z_1^2z_2) : z_1z_2^2z_3(z_3^2z_0 + z_1^2z_2) : \\
: z_2^3z_3(z_3^2z_0 + z_1^2z_2)] = [z_0 : z_1 : z_2 : z_3]
\]
wherever the composition $\psi \circ \varphi$ is well-defined. This shows that $\psi \circ \varphi$ is equivalent to the identity map on $S$.

Now let $[r : s : t]$ be a point in $\mathbb{P}^2$. Then we have
\[
(\varphi \circ \psi)([r : s : t]) = \varphi([rt(r^2s + t^3) : s^2(r^2s + t^3) : t^2(r^2s + t^3) : -t(r^2s + t^3)]) = [-r^2t^2(r^2s + t^3) : -s^2(r^2s + t^3) : t^2(r^2s + t^3) : -t(r^2s + t^3)] = [r : s : t]
\]
wherever the composition $\varphi \circ \psi$ is well-defined. This shows that $\varphi \circ \psi$ is equivalent to the identity map on $\mathbb{P}^2$. 

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