Solutions to Exercise Sheet 10

Solution 10.1. *Infinitely many lines on planes.*

(1) Since z_0 and $az_1 + bz_2 + cz_3$ are both homogeneous polynomials of degree 1 and not proportional to each other, $L = \mathbb{V}(z_0, az_1 + bz_2 + cz_3)$ defines a line in $\mathbb{P}^2$. To show that the line L is in P, we just need to observe that every point on L satisfies the equation $z_0 = 0$, hence is a point in P.

(2) Let $L = \mathbb{V}(z_0, az_1 + bz_2 + cz_3)$ and $L' = \mathbb{V}(z_0, a'z_1 + b'z_2 + c'z_3)$ be two such lines, where $[a : b : c] \neq [a' : b' : c']$. If a point $p = [z_0 : z_1 : z_2 : z_3]$ is an intersection point of L and L', then its coordinates satisfy the system of equations

$$\begin{aligned} z_0 &= 0; \\ az_1 + bz_2 + cz_3 &= 0; \\ a'z_1 + b'z_2 + c'z_3 &= 0. \end{aligned}$$

The first equation fixes the z_0 coordinate. For the other coordinates, we look at the second and the third equations. We look at the coefficient matrix

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}.$$

Since $[a : b : c]$ and $[a' : b' : c']$ represent different points in $\mathbb{P}^2$, both rows are non-zero and linearly independent. Hence the matrix has rank 2. It follows that the null-space has dimension 1, which means that there is a unique solution for $[z_1 : z_2 : z_3]$ (up to scaling). Therefore there is a unique intersection point $[z_0 : z_1 : z_2 : z_3]$ for the lines L and L'.

Solution 10.2. *Infinitely many lines on non-singular quadric surfaces.*

(1) It is clear that for every point $[a : b] \in \mathbb{P}^1$, the two polynomials $az_0 + bz_1$ and $az_2 + bz_3$ are non-zero and homogeneous of degree 1. They are not propotional to each other, so $\mathbb{V}(az_0 + bz_1, az_2 + bz_3)$ defines a line L in $\mathbb{P}^2$. We still need to show that every point in L is a point in Q. Since $[a : b] \in \mathbb{P}^1$, we have either $a \neq 0$ or $b \neq 0$. If $a \neq 0$, then a point $p = [z_0 : z_1 : z_2 : z_3] \in L$ satisfies $z_0 = -\frac{b}{a}z_1$ and $z_2 = -\frac{b}{a}z_3$. Then

$$z_0z_3 - z_1z_2 = \left(-\frac{b}{a}\right) \cdot z_1 \cdot z_3 - z_1 \cdot \left(-\frac{b}{a}\right) \cdot z_3 = 0.$$

Hence $p \in Q$. If $b \neq 0$, a similar calculation shows that every point $p \in L$ also satisfies the equation $z_0z_3 - z_1z_2 = 0$ hence is a point in Q. We conclude that L is a line in Q.

(2) Consider two lines $L = \mathbb{V}(az_0 + bz_1, az_2 + bz_3)$ and $L' = \mathbb{V}(a'z_0 + b'z_1, a'z_2 + b'z_3)$ where $[a : b]$ and $[a' : b']$ are two different points in $\mathbb{P}^1$. If the two lines have a common point $[z_0 : z_1 : z_2 : z_3]$, then the system of equations

$$\begin{aligned} az_0 + bz_1 &= 0, \\ az_2 + bz_3 &= 0, \\ a'z_0 + b'z_1 &= 0, \\ a'z_2 + b'z_3 &= 0 \end{aligned}$$

must have a non-zero solution. However, the coefficient matrix for the first and the third equations is $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$. Since $[a : b]$ and $[a' : b']$ are two different points in $\mathbb{P}^1$, the two rows are both non-zero and linearly independent. Hence the matrix has rank 2, which means that the only solution to these two equations is $z_0 = z_1 = 0$. For the same reason the only solution to the second and fourth equations is $z_2 = z_3 = 0$. Since the system of four equations has only a zero solution, L and L' do not have any common point. In other words, they are disjoint.

(3) For any point $p = [z_0 : z_1 : z_2 : z_3] \in Q$, we first show that p lies on a certain line $L = \mathbb{V}(az_0 + bz_1, az_2 + bz_3)$. There are two cases. *Case 1.* If z_0 and z_1 are not simultaneously zero, then we choose $[a : b] = [z_1 : -z_0]$ for the line L. We claim that $p \in L$. Indeed, for such a choice of $[a : b]$ we have $az_0 + bz_1 = z_1z_0 - z_0z_1 = 0$ and $az_2 + bz_3 = z_1z_2 - z_0z_3 = 0$. The claim holds. *Case 2.* If z_0 and z_1 are both zero, then z_2 and z_3 are not simultaneously zero. We can choose $[a : b] = [z_3 : -z_2]$ for the line L. A similar calculation shows that $p \in L$. In both cases, the point p lies on a certain line $L = \mathbb{V}(az_0 + bz_1, az_2 + bz_3)$ for a suitable choice of $[a : b]$.

It remains to prove that p lies on only one of such lines. This is clear because we have seen from part (2) that two such lines are always disjoint.

(4) For every $[a : b] \in \mathbb{P}^1$, $\mathbb{V}(az_0 + bz_2, az_1 + bz_3)$ also defines a line. These lines are pairwisely disjoint, and every point in Q lies on exactly one of them. The proof can be obtained simply by switching z_1 and z_2 in the proof for the above three parts.

Solution 10.3. *Rationality of a cubic surface.*

(1) We need to verify that every point in the image of ψ satisfies the defining equation of S. Indeed, we have

$$\begin{aligned}
&z_0^2z_1 + z_1^2z_2 + z_2^2z_3 + z_3^2z_0 \\
&= -r^2t^2(rt+s^2)^2 \cdot s(r^2s+t^3) + s^2(r^2s+t^3)^2 \cdot t^2(rt+s^2) \\
&\quad - t^4(rt+s^2)^2 \cdot t(r^2s+t^3) + t^2(r^2s+t^3)^2 \cdot rt(rt+s^2) \\
&= -(r^2t^2s+t^5)\cdot(rt+s^2)^2\cdot(r^2s+t^3) + (s^2t^2+rt^3)\cdot(r^2s+t^3)^2\cdot(rt+s^2) \\
&= -t^2(r^2s+t^3)\cdot(rt+s^2)^2\cdot(r^2s+t^3) + t^2(s^2+rt)\cdot(r^2s+t^3)^2\cdot(rt+s^2) \\
&= -t^2\cdot(rt+s^2)^2\cdot(r^2s+t^3)^2 + t^2\cdot(r^2s+t^3)^2\cdot(rt+s^2)^2 \\
&= 0.
\end{aligned}$$

Therefore the statement holds.

(2) Let $[z_0 : z_1 : z_2 : z_3]$ be a point in S. Then these coordinates satisfy

$$z_0^2z_1 + z_1^2z_2 + z_2^2z_3 + z_3^2z_0 = 0.$$

Then we have

$$\begin{aligned}
&(\psi\circ\varphi)([z_0 : z_1 : z_2 : z_3]) \\
&= \psi([z_0z_3 : z_1z_2 : z_2z_3]) \\
&= [z_0z_2z_3^2(z_0z_2z_3^2 + z_1^2z_2^2) : -z_1z_2(z_0^2z_1z_2z_3^2 + z_2^3z_3^3) : \\
&\qquad : z_2^2z_3^2(z_0z_2z_3^2 + z_1^2z_2^2) : -z_2z_3(z_0^2z_1z_2z_3^2 + z_2^3z_3^3)] \\
&= [z_0z_2^2z_3^2(z_3^2z_0 + z_1^2z_2) : -z_1z_2^2z_3^2(z_0^2z_1 + z_2^2z_3) : \\
&\qquad : z_2^3z_3^2(z_3^2z_0 + z_1^2z_2) : -z_2^2z_3^3(z_0^2z_1 + z_2^2z_3)] \\
&= [z_0z_2^2z_3^2(z_3^2z_0 + z_1^2z_2) : z_1z_2^2z_3^2(z_3^2z_0 + z_1^2z_2) : \\
&\qquad : z_2^3z_3^2(z_3^2z_0 + z_1^2z_2) : z_2^2z_3^3(z_3^2z_0 + z_1^2z_2)] \\
&= [z_0 : z_1 : z_2 : z_3]
\end{aligned}$$

wherever the composition $\psi\circ\varphi$ is well-defined. This shows that $\psi\circ\varphi$ is equivalent to the identity map on S.

Now let $[r : s : t]$ be a point in $\mathbb{P}^2$. Then we have

$$\begin{aligned}
&(\varphi\circ\psi)([r : s : t]) \\
&= \varphi([rt(rt+s^2) : -s(r^2s+t^3) : t^2(rt+s^2) : -t(r^2s+t^3)]) \\
&= [-rt^2(rt+s^2)(r^2s+t^3) : -st^2(r^2s+t^3)(rt+s^2) : -t^3(rt+s^2)(r^2s+t^3)] \\
&= [r : s : t]
\end{aligned}$$

wherever the composition $\varphi\circ\psi$ is well-defined. This shows that $\varphi\circ\psi$ is equivalent to the identity map on $\mathbb{P}^2$.